Denotational semantics of PCF

Proposition. *For all typing judgements* $\Gamma \vdash M : \tau$, *the denotation*

$$[\![\Gamma \vdash M]\!] : [\![\Gamma]\!] \to [\![\tau]\!]$$

is a well-defined continous function.

Denotations of closed terms

For a closed term $M \in \mathrm{PCF}_\tau$, we get

$$[\![\emptyset \vdash M]\!] : [\![\emptyset]\!] \to [\![\tau]\!]$$

and, since $[\![\emptyset]\!] = \{\perp\}$, we have

$$\boxed{[\![M]\!] \stackrel{\text{def}}{=} [\![\emptyset \vdash M]\!](\perp) \in [\![\tau]\!] \qquad (M \in \mathrm{PCF}_\tau)}$$

Compositionality

Proposition. *For all typing judgements* $\Gamma \vdash M : \tau$ *and* $\Gamma \vdash M' : \tau$, *and all contexts* $\mathcal{C}[-]$ *such that* $\Gamma' \vdash \mathcal{C}[M] : \tau'$ *and* $\Gamma' \vdash \mathcal{C}[M'] : \tau'$,

if $[\![\Gamma \vdash M]\!] = [\![\Gamma \vdash M']\!] : [\![\Gamma]\!] \to [\![\tau]\!]$

then $[\![\Gamma' \vdash \mathcal{C}[M]]\!] = [\![\Gamma' \vdash \mathcal{C}[M]]\!] : [\![\Gamma']\!] \to [\![\tau']\!]$

Proof idea:

Consider $C[\,]$ of the form $[\,]\,N$

RTP: $[\![M]\!] = [\![M']\!] \overset{?}{\Longrightarrow} [\![M\,N]\!] = [\![M'\,N]\!]$

$$[\![M\,N]\!]\rho = ([\![M]\!]\rho)([\![N]\!]\rho)$$

$$= ([\![M']\!]\rho)([\![N]\!]\rho)$$

$$= [\![M'\,N]\!]\rho.$$

Soundness

Proposition. *For all closed terms* $M, V \in \mathrm{PCF}_\tau$*,*

$$\textit{if } M \Downarrow_\tau V \textit{ then } [\![M]\!] = [\![V]\!] \in [\![\tau]\!] \;.$$

Proof idea:

Case

$$\frac{M \Downarrow \underline{succ}(V)}{\underline{pred}(M) \Downarrow V}$$

RTP $[\![\underline{pred}(M)]\!] \stackrel{?}{=} [\![V]\!]$

By ind. $[\![M]\!] = [\![\underline{succ}\ V]\!] = [\![V]\!] + 1$

Then $[\![\underline{pred}\ (M)]\!] = p([\![M]\!]) = p([\![V]\!] + 1) = [\![V]\!]$

Case

$$\frac{M\,(\mathrm{fix}\,M) \Downarrow V}{\underline{\mathrm{fix}}(M) \Downarrow V}$$

By ind

$$[\![M\,(\mathrm{fix}\,M)]\!] = [\![V]\!]$$

$$\begin{aligned}
&\quad \shortparallel \\
&[\![M]\!]\,([\![\mathrm{fix}\,M]\!]) \\
&\quad \shortparallel \\
&[\![M]\!]\,(\underline{\mathrm{fix}}\,[\![M]\!]) \\
&\quad \shortparallel \\
&\underline{\mathrm{fix}}\,[\![M]\!] \\
&\quad \shortparallel \\
&[\![\mathrm{fix}\,M]\!]
\end{aligned}$$

Case

$$\frac{M_1 \Downarrow \mathrm{fn}\, x.\, M \qquad M[M_2/x] \Downarrow V}{M_1\, M_2 \Downarrow V}$$

RTP: $[\![M_1\, M_2]\!] \stackrel{?}{=} [\![V]\!]$

$\quad\; \|$

$[\![M_1]\!] ([\![M_2]\!])$

By induction: $[\![M_1]\!] = [\![\mathrm{fn}\, x.\, M]\!]$

$[\![M[M_2/x]]\!] = [\![V]\!]$

So $[\![M_1]\!]([\![M_2]\!]) = [\![\mathrm{fn}\, x.\, M]\!]([\![M_2]\!]) =$

$$\llbracket \mathrm{fn}\, x.\, M \rrbracket \, (\llbracket M_2 \rrbracket)$$

$$= (\lambda d.\, \llbracket M \rrbracket [x \mapsto d])\, \llbracket M_2 \rrbracket$$

$$= \llbracket M \rrbracket [x \mapsto \llbracket M_2 \rrbracket]$$

$$\stackrel{?}{=} \llbracket M[{}^{M_2}/_{x}] \rrbracket$$

$$= \llbracket V \rrbracket$$

Lemma
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Substitution lemma

Substitution property

Proposition. *Suppose that* $\Gamma \vdash M : \tau$ *and that* $\Gamma[x \mapsto \tau] \vdash M' : \tau'$, *so that we also have* $\Gamma \vdash M'[M/x] : \tau'$.

Then,

$$\llbracket \Gamma \vdash M'[M/x] \rrbracket (\rho)$$
$$= \llbracket \Gamma[x \mapsto \tau] \vdash M' \rrbracket \big(\rho\big[x \mapsto \llbracket \Gamma \vdash M \rrbracket(\rho)\big]\big)$$

for all $\rho \in \llbracket \Gamma \rrbracket$.

Consider the case $M' = \mathbf{fn}\, y : \sigma.\, N$ with $\tau' = \sigma \to \sigma'$

$$\llbracket \Gamma \vdash (\mathbf{fn}\, y.\, N)[M/x] \rrbracket (\rho)$$
$$= \llbracket \Gamma \vdash \mathbf{fn}\, y.\, (N[M/x]) \rrbracket (\rho)$$

$$[\![\Gamma \vdash \mathrm{fn}\, y.\, N[M/x]]\!](\rho) \qquad \rho \in [\![\Gamma]\!]$$

$$= \lambda d.\, [\![\Gamma[y \mapsto \sigma] \vdash N[M/x]]\!](\rho[y \mapsto d])$$

ind

$$= \lambda d.\, \underline{[\![\Gamma[y \mapsto \sigma][x \mapsto \tau] \vdash N]\!]}\, (\rho[y \mapsto d][x \mapsto \underline{[\![M]\!](\rho[y \mapsto d])}])$$

$$[\![\Gamma[x \mapsto \tau] \vdash \mathrm{fn}\, y.\, N]\!](\rho[x \mapsto [\![M]\!](\rho)])$$

$$= \lambda d.\, \underline{[\![\Gamma[x \mapsto \tau][y \mapsto \sigma] \vdash N]\!]}\, (\rho[x \mapsto \underline{[\![M]\!](\rho)}, y \mapsto d])$$

Substitution property

Proposition. *Suppose that* $\Gamma \vdash M : \tau$ *and that* $\Gamma[x \mapsto \tau] \vdash M' : \tau'$*, so that we also have* $\Gamma \vdash M'[M/x] : \tau'$*. Then,*

$$\begin{aligned} &[\![\Gamma \vdash M'[M/x]]\!](\rho) \\ &\qquad = [\![\Gamma[x \mapsto \tau] \vdash M']\!]\big(\rho\big[x \mapsto [\![\Gamma \vdash M]\!](\rho)\big]\big) \end{aligned}$$

for all $\rho \in [\![\Gamma]\!]$*.*

In particular when $\Gamma = \emptyset$, $[\![\langle x \mapsto \tau\rangle \vdash M']\!] : [\![\tau]\!] \to [\![\tau']\!]$ and

$$[\![M'[M/x]]\!] = [\![\langle x \mapsto \tau\rangle \vdash M']\!]([\![M]\!])$$

Weakening Property

Proposition Suppose $\Gamma \vdash M : \tau$.
Then, for $y \notin \underline{\mathrm{dom}}(\Gamma)$,

$$[\![\Gamma[y \mapsto \sigma] \vdash M]\!](\rho[y \mapsto d])$$

$$= [\![\Gamma \vdash M]\!](\rho)$$

for all $\rho \in [\![\Gamma]\!]$ and $d \in [\![\sigma]\!]$.

NB: One proves

- a weakening lemma

to prove

- a substitution lemma

to prove

- denotational soundness.

General Denotational Semantics.

$D_1 \times \cdots \times D_n \to D_i$ proj_i

projections

- Domains: $[\![\mathrm{nat}]\!]$ $[\![\mathrm{bool}]\!]$.
- Supporting products ($\times$) and function ($\to$) constructions.
- Interpretations for basic functions (pred, succ, ...)
- $\mathrm{fix} : (D \to D) \to D$

$$(D \to E) \times D \xrightarrow{ev} E$$

$$\frac{D \times E \to F}{D \to (E \to F)} \text{ (curry)}$$

Example: Stable domains.