# *Topic 4*

Scott Induction

$f : D \to D$ continuous

$S \subseteq D$ ~ $d \in S \iff$ it satisfies the property of interest.

$\underline{fix}(f) \in S$ ?

(1) $\perp \in S$

(2) <u>Invariance</u>:

$$x \in S \Rightarrow f(x) \in S$$

Then

$$\perp \in S$$

$$f(\perp) \in S$$

$$\vdots$$

$$f^n(\perp) \in S$$

We have the chain

$$\perp \sqsubseteq f(\perp) \sqsubseteq \cdots \sqsubseteq f^n(\perp) \sqsubseteq \cdots \quad \text{in } S$$

(3) Let $S$ be <u>chain closed</u>

If $d_0 \sqsubseteq d_1 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \cdots$ is in $S$ then $\bigsqcup_n d_n \in S$

Then

$$\mathit{fix}(f) = \bigsqcup_n f^n(\bot) \in S$$

$$\frac{\forall x \in D.\ x \in S \Rightarrow f(x) \in S}{\underline{\mathit{fix}}(f) \in S} \ (S \text{ admissible})$$

def

$\bot \in S$

& chain closed.

## Scott's Fixed Point Induction Principle

Let $f : D \to D$ be a continuous function on a domain $D$.

For any admissible subset $S \subseteq D$, to prove that the least fixed point of $f$ is in $S$, *i.e.* that

$$\mathit{fix}(f) \in S \ ,$$

it suffices to prove

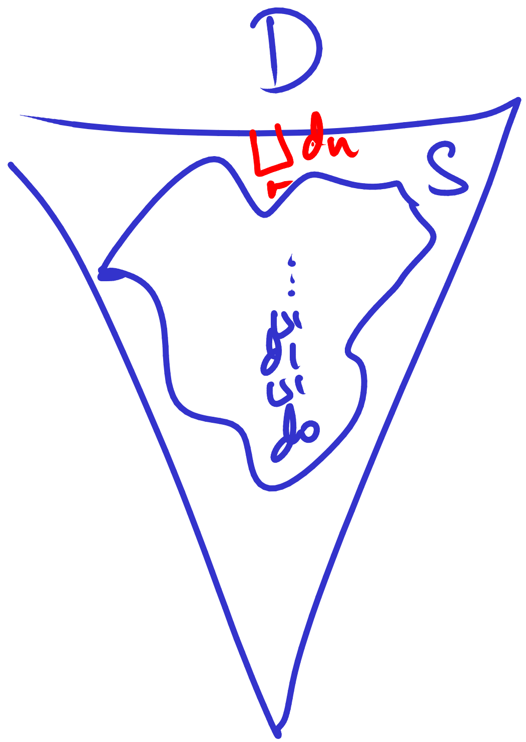
$$\forall d \in D \ (d \in S \ \Rightarrow \ f(d) \in S) \ .$$

# Chain-closed and admissible subsets

Let $D$ be a cpo. A subset $S \subseteq D$ is called chain-closed iff for all chains $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots$ in $D$

$$(\forall n \geq 0 \,.\, d_n \in S) \;\Rightarrow\; \left( \bigsqcup_{n \geq 0} d_n \right) \in S$$

If $D$ is a domain, $S \subseteq D$ is called admissible iff it is a chain-closed subset of $D$ and $\perp \in S$.



## Chain-closed and admissible subsets

Let $D$ be a cpo. A subset $S \subseteq D$ is called chain-closed iff for all chains $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots$ in $D$

$$(\forall n \geq 0 \,.\, d_n \in S) \;\Rightarrow\; \Big(\bigsqcup_{n \geq 0} d_n\Big) \in S$$

If $D$ is a domain, $S \subseteq D$ is called admissible iff it is a chain-closed subset of $D$ and $\bot \in S$.

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A property $\Phi(d)$ of elements $d \in D$ is called *chain-closed* (resp. *admissible*) iff $\{d \in D \mid \Phi(d)\}$ is a *chain-closed* (resp. *admissible*) subset of $D$.

# Building chain-closed subsets (I)

Let $D, E$ be cpos.

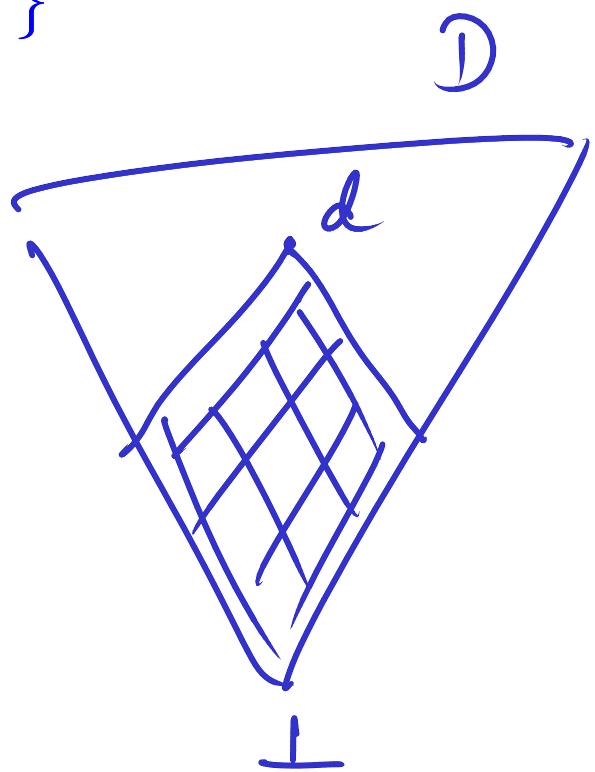
**Basic relations:**

- For every $d \in D$, the subset

downset of $d \in$

$$\downarrow(d) \stackrel{\text{def}}{=} \{ x \in D \mid x \sqsubseteq d \}$$

of $D$ is chain-closed.



## Building chain-closed subsets (I)

$\bigsqcup_n (x_n, y_n) \in \sqsubseteq$

Let $D, E$ be cpos.

$(x_0, y_0) \sqsubseteq (x_1, y_1) \sqsubseteq \cdots$

iff

$x_0 \sqsubseteq x_1 \sqsubseteq \cdots$ in $D$

$y_0 \sqsubseteq y_1 \sqsubseteq \cdots$ in $D$

**Basic relations:**

- For every $d \in D$, the subset

$$\downarrow(d) \stackrel{\text{def}}{=} \{ x \in D \mid x \sqsubseteq d \}$$

$\bigsqcup_n x_n \sqsubseteq \bigsqcup_n y_n$

of $D$ is chain-closed.

$\sqsubseteq \subseteq D \times D$

- The subsets

$$\{(x, y) \in D \times D \mid x \sqsubseteq y\}$$

and

$$\{(x, y) \in D \times D \mid x = y\}$$

of $D \times D$ are chain-closed.

# Example (I): Least pre-fixed point property

Let $D$ be a domain and let $f : D \to D$ be a continuous function.

$$\forall d \in D.\, f(d) \sqsubseteq d \implies \mathit{fix}(f) \sqsubseteq d$$

$$f(d) \sqsubseteq d$$

$$\frac{x \sqsubseteq d}{f x \sqsubseteq f(d)}$$

$$\Downarrow$$

$$\frac{f(x) \sqsubseteq d}{x \sqsubseteq d \Rightarrow f(x) \sqsubseteq d}$$

$$\frac{x \in \downarrow(d) \Rightarrow f(x) \in \downarrow(d)}{\mathit{fix}(f) \sqsubseteq d \iff \mathit{fix}(f) \in \downarrow(d)}$$

## Example (I): Least pre-fixed point property

Let $D$ be a domain and let $f : D \to D$ be a continuous function.

$$\forall d \in D.\, f(d) \sqsubseteq d \implies \mathit{fix}(f) \sqsubseteq d$$

*Proof by Scott induction.*

Let $d \in D$ be a pre-fixed point of $f$. Then,

$$\begin{aligned} x \in {\downarrow}(d) \quad & \implies \quad x \sqsubseteq d \\ & \implies \quad f(x) \sqsubseteq f(d) \\ & \implies \quad f(x) \sqsubseteq d \\ & \implies \quad f(x) \in {\downarrow}(d) \end{aligned}$$

Hence,

$$\mathit{fix}(f) \in {\downarrow}(d) \ .$$

$$\{d \in D \mid f(d) \in S\} = f^{-1}(S)$$

$$\begin{array}{ccc} f^{-1}(S) & & S \\ \cap\!\!\mid & & \cap\!\!\mid \\ D & \xrightarrow{f} & E \end{array}$$

## Building chain-closed subsets (II)

**Inverse image:**

Let $f : D \to E$ be a continuous function.

If $S$ is a chain-closed subset of $E$ then the inverse image

$$f^{-1}S \;=\; \{x \in D \mid f(x) \in S\}$$

is an chain-closed subset of $D$.

$\bigsqcup_n d_n \in f^{-1}(S)$

$f(\bigsqcup_n d_n) = \bigsqcup_n f(d_n)$ in $S$

in $f^{-1}(S)$: $d_0 \sqsubseteq d_1 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \cdots$

$f(d_0) \sqsubseteq f(d_1) \sqsubseteq \cdots \sqsubseteq f(d_n) \sqsubseteq \cdots$

## Example (II)

Let $D$ be a domain and let $f, g : D \to D$ be continuous functions such that $f \circ g \sqsubseteq g \circ f$. Then,

$$f(\bot) \sqsubseteq g(\bot) \implies \mathit{fix}(f) \sqsubseteq \mathit{fix}(g) \ .$$

$$\frac{x \sqsubseteq \mathit{fix}(g)}{f(x) \sqsubseteq f(\mathit{fix}\, g)} \qquad \frac{\boxed{f(\mathit{fix}\, g) \sqsubseteq g(\mathit{fix}\, g)}}{f(\mathit{fix}\, g) \sqsubseteq \mathit{fix}(g)}$$

$$\frac{\dfrac{f(x) \sqsubseteq \mathit{fix}(g)}{x \sqsubseteq \mathit{fix}(g) \Rightarrow f(x) \sqsubseteq \mathit{fix}(g)}}{\mathit{fix}(f) \sqsubseteq \mathit{fix}(g)} \ \downarrow(\mathit{fix}(g))$$

$$\langle f,g\rangle : D \to D\times D$$
$$d \mapsto (f(d), g(d))$$

$\Pi A$ $\quad \perp \in \langle f,g\rangle^{-1}(\sqsubseteq)$

$f(\perp) \sqsubseteq g(\perp)$ ✓

$$\dfrac{f(x)\sqsubseteq g(x)}{g(f x)\sqsubseteq g\,g\,x}$$

$\| \,?$

$$\dfrac{f(g x)\sqsubseteq g(g\,x)}{\phantom{x}}$$

$$\dfrac{x\in\langle f,g\rangle^{-1}(\sqsubseteq) \implies g(x)\in\langle f,g\rangle^{-1}\sqsubseteq}{f(\mathrm{fix}\,g)\sqsubseteq g(\mathrm{fix}\,g) \iff \mathrm{fix}(g)\in\langle f,g\rangle^{-1}(\sqsubseteq)}$$

$\langle f,g\rangle^{-1}(\sqsubseteq)$ chain closed $\Longleftarrow$ $\sqsubseteq$ chain closed

$\{x \mid f x \sqsubseteq g x\}$

admiss.

$\lesssim$

## Example (II)

Let $D$ be a domain and let $f, g : D \to D$ be continuous functions such that $f \circ g \sqsubseteq g \circ f$. Then,

$$f(\bot) \sqsubseteq g(\bot) \implies \mathit{fix}(f) \sqsubseteq \mathit{fix}(g) \ .$$

*Proof by Scott induction.*

Consider the admissible property $\Phi(x) \equiv \big(f(x) \sqsubseteq g(x)\big)$ of $D$.

Since

$$f(x) \sqsubseteq g(x) \Rightarrow g(f(x)) \sqsubseteq g(g(x)) \Rightarrow f(g(x)) \sqsubseteq g(g(x))$$

we have that

$$f(\mathit{fix}(g)) \sqsubseteq g(\mathit{fix}(g)) \ .$$

# Building chain-closed subsets (III)

**Logical operations:**

- If $S, T \subseteq D$ are chain-closed subsets of $D$ then

  $$S \cup T \qquad \text{and} \qquad S \cap T$$

  are chain-closed subsets of $D$.

- If $\{ S_i \}_{i \in I}$ is a family of chain-closed subsets of $D$ indexed by a set $I$, then $\bigcap_{i \in I} S_i$ is a chain-closed subset of $D$.

- If a property $P(x, y)$ determines a chain-closed subset of $D \times E$, then the property $\forall x \in D.\ P(x, y)$ determines a chain-closed subset of $E$.

Consider $(\mathbb{N})$

$$\perp \sqsubseteq 0 \sqsubseteq 1 \sqsubseteq \cdots \sqsubseteq n \sqsubseteq \cdots \sqsubseteq \infty$$

For every $n \in \mathbb{N}$

$\downarrow(n)$ is admissible

but

$$\bigcup_n \downarrow(n) = \{\perp, 0, 1, \ldots, n, \ldots \mid n \in \mathbb{N}\}$$

is not chain closed.

If $S$ and $T$ are chain closed then so is $S \cup T$.

$d_0 \sqsubseteq d_1 \sqsubseteq \dots \sqsubseteq d_n \sqsubseteq \dots$ in $S \cup T$.

RTP: $(\bigsqcup_n d_n) \in S \cup T$.

Every $d_i$ is either in $S$ or in $T$.

$d_{s(0)} \sqsubseteq d_{s(1)} \sqsubseteq d_{s(2)} \sqsubseteq \dots$ in $S$

$d_{t(0)} \sqsubseteq d_{t(1)} \sqsubseteq d_{t(2)} \sqsubseteq \dots$ in $T$

Say the chain of $d_{s(i)}$ is infinite. Then $(\bigsqcup_i d_{s(i)}) \in S$

and $(\bigsqcup_i d_{s(i)}) \in S \cup T$

Let $d_0 \sqsubseteq d_1 \sqsubseteq \dots \sqsubseteq d_n \sqsubseteq \dots$ in $D$

and $e_0 \sqsubseteq e_1 \sqsubseteq \dots \sqsubseteq e_n \sqsubseteq \dots$

~~Exercise~~

$$\forall i \,.\, \exists j \,.\, d_i \sqsubseteq e_j \implies \bigsqcup_i d_i \sqsubseteq \bigsqcup_i e_i$$

## Example (III): Partial correctness

Let $\mathcal{F} : State \rightharpoonup State$ be the denotation of

$$\textbf{while}\ X > 0\ \textbf{do}\ (Y := X * Y; X := X - 1)\ .$$

For all $x, y \geq 0$,

$$\mathcal{F}[X \mapsto x, Y \mapsto y] \downarrow$$
$$\implies \mathcal{F}[X \mapsto x, Y \mapsto y] = [X \mapsto 0, Y \mapsto x! \cdot y].$$

location X stores x
Y stores y $\rightsquigarrow$ or $(x, y)$

Recall that

$$\mathcal{F} = \mathit{fix}(f)$$

where $f : (State \rightharpoonup State) \to (State \rightharpoonup State)$ is given by

$$f(w) = \lambda(x, y) \in State. \begin{cases} (x, y) & \text{if } x \leq 0 \\ w(x - 1, x \cdot y) & \text{if } x > 0 \end{cases}$$

*Proof by Scott induction.*

We consider the admissible subset of $(State \rightharpoonup State)$ given by

$$S = \left\{ w \;\middle|\; \begin{array}{l} \forall x, y \geq 0. \\ \quad w[X \mapsto x, Y \mapsto y]\downarrow \\ \qquad \Rightarrow w[X \mapsto x, Y \mapsto y] = [X \mapsto 0, Y \mapsto x! \cdot y] \end{array} \right\}$$

and show that

$$w \in S \implies f(w) \in S \ .$$