Continuity and strictness

- If D and E are cpo's, the function f is continuous iff

  1. it is monotone, and
  2. it preserves lubs of chains, *i.e.* for all chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots$ in D, it is the case that

$$f\Big(\bigsqcup_{n \geq 0} d_n\Big) = \bigsqcup_{n \geq 0} f(d_n) \quad \text{in } E.$$

- If D and E have least elements, then the function f is strict iff $f(\perp) = \perp$.

Domain of streams

Stream($\mathbb{N}$) elements $n_0, n_1, n_2, \ldots$

$n_k = \bot \Rightarrow n_{k+1} = \bot$

order $\vec{n} \sqsubseteq \vec{m}$

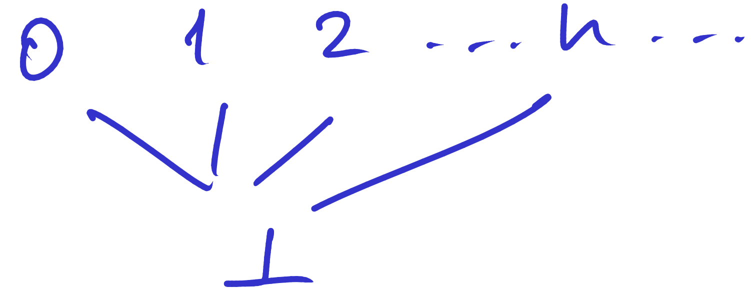
iff$_{def}$ $\forall i.\ n_i \sqsubseteq m_i$

lubs $\vec{n_0} \sqsubseteq \vec{n_1} \sqsubseteq \vec{n_2} \sqsubseteq \ldots \sqsubseteq \vec{n_k} \sqsubseteq$

$$\left(\bigsqcup_k \vec{n_k}\right)_i = \bigsqcup_k (n_k)_i$$

bottom $\vec{\bot} = \bot, \bot, \ldots, \bot, \ldots$

$$f : \underline{Stream(\mathbb{N})} \longrightarrow \mathbb{N}_\perp$$



$k \in \mathbb{N}$

$\parallel$

$$f(n_0, n_1, n_2, \dots, n_i, \dots)$$

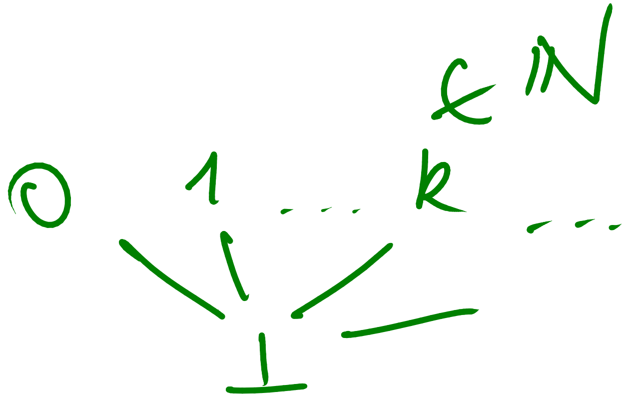
$$= f\Big(\bigsqcup\big[(\perp,\perp,\perp,\dots,\perp,\dots) \sqsubseteq (n_0,\perp,\perp,\dots,\perp) \sqsubseteq (n_0,n_1,\perp,\dots,\perp) \sqsubseteq (n_0,n_1,n_2,\perp,\dots,\perp,\dots) \sqsubseteq \dots\big]\Big)$$

$$= \bigsqcup\big[f(\perp,\perp,\dots,\perp) \sqsubseteq f(n_0,\perp,\perp,\dots,\perp,\dots) \sqsubseteq f(n_0,n_1,\perp,\dots) \sqsubseteq f(n_0,n_1,n_2,\perp,\dots,\perp,\dots) \sqsubseteq \dots\big]$$

$$k = \bigsqcup \big[\, f(\bot\ \bot \dots),\ f(n_0\ \bot \dots \bot \dots),\ f(n_0\ n_1\ \bot \dots \bot \dots),\ \dots \,\big]$$

$$\Downarrow$$

$$f(n_0\, n_1 \dots n_i\ \bot \dots \bot \dots) \in \{\bot, k\}$$



$$\Downarrow$$

$$f(n_0\, n_1 \dots n_l\ \bot \dots \bot \dots) = k \text{ for some } l \in \mathbb{N}.$$

Tarski's Fixed Point Theorem

Let $f : D \to D$ be a continuous function on a domain D. Then

- f possesses a least pre-fixed point, given by

$$\mathit{fix}(f) = \bigsqcup_{n \geq 0} f^n(\perp).$$

- Moreover, $\mathit{fix}(f)$ is a fixed point of f, *i.e.* satisfies $f\big(\mathit{fix}(f)\big) = \mathit{fix}(f)$, and hence is the least fixed point of f.

$f : D \to D$ continuous

$$\perp \sqsubseteq f(\perp) \underset{\text{mon}}{\Longrightarrow} f(\perp) \sqsubseteq f^2(\perp)$$

$$\Rightarrow \perp \sqsubseteq f(\perp) \sqsubseteq f^2(\perp) \sqsubseteq \cdots \sqsubseteq f^n(\perp) \sqsubseteq \cdots \text{ in } D$$

$$\bigsqcup(\perp \sqsubseteq f(\perp) \sqsubseteq \cdots \sqsubseteq f^n(\perp) \sqsubseteq \cdots)$$

$= \bigsqcup_n f^n(\perp)$ is a fixed point.

$$f\left(\bigsqcup_n f^n(\perp)\right) = \bigsqcup_n f\left(f^n(\perp)\right) = \bigsqcup_n f^{n+1}(\perp)$$

$$= \bigsqcup(f(\perp) \sqsubseteq f^2(\perp) \sqsubseteq \cdots \sqsubseteq f^n(\perp) \sqsubseteq \cdots) = \bigsqcup_n f^n(\perp)$$

LEAST

$$f(d) \sqsubseteq d \Rightarrow \bigsqcup_n f^n(\bot) \sqsubseteq d$$

$\bot \sqsubseteq d$ ✓ $\Rightarrow$ $f(\bot) \sqsubseteq f(d) \sqsubseteq d$

$f(\bot) \sqsubseteq d$ ✓

$f^2(\bot) \sqsubseteq d$

$\vdots$

$$\frac{\checkmark}{\frac{\forall n \quad f^n(\bot) \sqsubseteq d}{\bigsqcup_n f^n(\bot) \sqsubseteq d}}$$

$[\![\textbf{while } B \textbf{ do } C]\!]$: $\underline{State} \to \underline{State}$

$$[\![\textbf{while } B \textbf{ do } C]\!]$$

$$= \mathit{fix}(f_{[\![B]\!],[\![C]\!]})$$

$$= \bigsqcup_{n \geq 0} f_{[\![B]\!],[\![C]\!]}{}^{n}(\bot)$$

$$= \quad \lambda s \in \mathit{State}.$$

$$\begin{cases} [\![C]\!]^k(s) & \text{if } k \geq 0 \text{ is such that } [\![B]\!]([\![C]\!]^k(s)) = \mathit{false} \\ & \text{and } [\![B]\!]([\![C]\!]^i(s)) = \mathit{true} \text{ for all } 0 \leq i < k \\ \\ \text{undefined} & \text{if } [\![B]\!]([\![C]\!]^i(s)) = \mathit{true} \text{ for all } i \geq 0 \end{cases}$$

$f_{[\![B]\!],[\![C]\!]} : (\underline{State} \to \underline{State}) \to (\underline{State} \to \underline{State})$

continuous.

Topic 3

Constructions on Domains

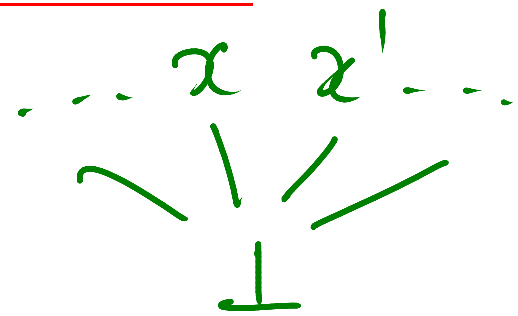
Models for datatypes

$\alpha * \beta$

$\alpha \to \beta$

Discrete cpo's and flat domains

For any set X, the relation of equality



$$x \sqsubseteq x' \overset{\text{def}}{\Leftrightarrow} x = x' \qquad (x, x' \in X)$$

makes $(X, \sqsubseteq)$ into a cpo, called the discrete cpo with underlying set X.

Let $X_\perp \overset{\text{def}}{=} X \cup \{\perp\}$, where $\perp$ is some element not in X. Then

$$d \sqsubseteq d' \overset{\text{def}}{\Leftrightarrow} (d = d') \vee (d = \perp) \qquad (d, d' \in X_\perp)$$

makes $(X_\perp, \sqsubseteq)$ into a domain (with least element $\perp$), called the flat domain determined by X.

<u>Example</u> $\mathbb{N}_\perp$, $\{true, false\}_\perp$

Binary product of cpo's and domains

The product of two cpo's $(D_1, \sqsubseteq_1)$ and $(D_2, \sqsubseteq_2)$ has underlying set

$$D_1 \times D_2 = \{(d_1, d_2) \mid d_1 \in D_1 \ \&\ d_2 \in D_2\}$$

and partial order $\sqsubseteq$ defined by

$$(d_1, d_2) \sqsubseteq (d_1', d_2') \ \stackrel{\text{def}}{\Leftrightarrow}\ d_1 \sqsubseteq_1 d_1' \ \&\ d_2 \sqsubseteq_2 d_2' \ .$$

$$\dfrac{(x_1, x_2) \sqsubseteq (y_1, y_2)}{\dfrac{}{x_1 \sqsubseteq_1 y_1 \qquad x_2 \sqsubseteq_2 y_2}}$$

Given D_1, D_2 domains

turn $D_1 \times D_2 = \{ (d_1, d_2) \mid d_1 \in D_1,\ d_2 \in D_2 \}$

into a domain

$(x_1, x_2) \sqsubseteq_{D_1 \times D_2} (y_1, y_2)$ is a partial order

iff$_{def}$ $x_1 \sqsubseteq_{D_1} y_1$ and $x_2 \sqsubseteq_{D_2} y_2$

$\perp_{D_1 \times D_2} = (\perp_{D_1}, \perp_{D_2})$

$(x_0, y_0) \sqsubseteq (x_1, y_1) \sqsubseteq \cdots \sqsubseteq (x_n, y_n) \sqsubseteq \cdots$ in $D_1 \times D_2$

iff

$x_0 \sqsubseteq x_1 \sqsubseteq \cdots \sqsubseteq x_n \sqsubseteq \cdots$ in D_1

and $y_0 \sqsubseteq y_1 \sqsubseteq \cdots \sqsubseteq y_n \sqsubseteq \cdots$ in D_2

Consider $\bigsqcup_n x_n$ in D_1 and $\bigsqcup_m y_m$ in D_2

so that $\left(\bigsqcup_n x_n, \bigsqcup_m y_m\right) \in D_1 \times D_2$

Claim

$$\left(\bigsqcup_n x_n, \bigsqcup_m y_m\right) = \bigsqcup_k (x_k, y_k)$$

Lubs of chains are calculated componentwise:

$$\bigsqcup_{n \geq 0} (d_{1,n}, d_{2,n}) = (\bigsqcup_{i \geq 0} d_{1,i}, \bigsqcup_{j \geq 0} d_{2,j}) \ .$$

If $(D_1, \sqsubseteq_1)$ and $(D_2, \sqsubseteq_2)$ are domains so is $(D_1 \times D_2, \sqsubseteq)$ and $\perp_{D_1 \times D_2} = (\perp_{D_1}, \perp_{D_2})$.

Continuous functions of two arguments

Proposition. *Let* D, E, F *be cpo's. A function* $f : (D \times E) \to F$ *is monotone if and only if it is monotone in each argument separately:*

$$\forall d, d' \in D, e \in E.\, d \sqsubseteq d' \;\Rightarrow\; f(d,e) \sqsubseteq f(d',e)$$
$$\forall d \in D, e, e' \in E.\, e \sqsubseteq e' \;\Rightarrow\; f(d,e) \sqsubseteq f(d,e').$$

Moreover, it is continuous if and only if it preserves lubs of chains in each argument separately:

$$f(\bigsqcup_{m \geq 0} d_m\, ,\, e) = \bigsqcup_{m \geq 0} f(d_m, e)$$
$$f(d\, ,\, \bigsqcup_{n \geq 0} e_n) = \bigsqcup_{n \geq 0} f(d, e_n).$$

$f : (D \times E) \to F$ for D, E, F domains.

continuous

iff

$f(d,-) : E \to F$ $\quad d \in D$

$e \mapsto f(d,e)$

and

$f(-,e) : D \to F$ $\quad e \in E$

$d \mapsto f(d,e)$

continuous.

NB (1) $f\left(\bigsqcup_m (x_m, y_m)\right) = \bigsqcup_m f(x_m, y_m)$

$\quad\| $

$f\left(\bigsqcup_m x_m, \bigsqcup_m y_m\right)$

- A couple of derived rules:

$$\frac{x \sqsubseteq x' \qquad y \sqsubseteq y'}{f(x,y) \sqsubseteq f(x',y')} \quad (f \text{ monotone})$$

f cont in the 1st arg

$$\frac{}{f(\bigsqcup_m x_m, \bigsqcup_n y_n) = \bigsqcup_k f(x_k, y_k)} \quad (f \text{ cont})$$

cont in 2nd arg

$f\left(\bigsqcup_m x_m, \bigsqcup_n y_n\right) = \bigsqcup_m f\left(x_m, \bigsqcup_n y_n\right)$

$= \bigsqcup_m \bigsqcup_n f(x_m, y_n)$

$= \bigsqcup_k f(x_k, y_k)$

Function cpo's and domains

Given cpo's $(D, \sqsubseteq_D)$ and $(E, \sqsubseteq_E)$, the function cpo $(D \to E, \sqsubseteq)$ has underlying set

$$(D \to E) \stackrel{\text{def}}{=} \{f \mid f : D \to E \text{ is a } \textit{continuous} \text{ function}\}$$

and partial order: $f \sqsubseteq f' \stackrel{\text{def}}{\Leftrightarrow} \forall d \in D \,.\, f(d) \sqsubseteq_E f'(d)$.

$$\left\{\begin{array}{l} f \sqsubseteq f \\ f \sqsubseteq g \wedge g \sqsubseteq f \Rightarrow f = g \\ f \sqsubseteq g \wedge g \sqsubseteq h \Rightarrow f \sqsubseteq h \end{array}\right.$$

$\bot_{(D \to E)} = \lambda d \in D \,.\, \bot_E$ is continuous.

$f_0 \sqsubseteq f_1 \sqsubseteq \cdots \sqsubseteq f_n \sqsubseteq \cdots$ in $(D \to E)$

$f_0(d) \sqsubseteq f_1(d) \sqsubseteq \cdots$

in E

Define $\bigsqcup_n f_n \in (D \to E)$

?

$$\left(\bigsqcup_n f_n\right)(d) \stackrel{\text{def}}{=} \bigsqcup_n \left(f_n(d)\right)$$

RTP $\bigsqcup_n f_n$ is continous from D to E.

$$\left(\bigsqcup_n f_n\right)(x) \sqsubseteq \left(\bigsqcup_n f_n\right)(y) \quad \forall\, x \sqsubseteq y$$

Exercise

$\wedge$

$$\left(\bigsqcup_n f_n\right)\left(\bigsqcup_k x_k\right) = \bigsqcup_k \left(\bigsqcup_n f_n\right)(x_k)$$