Thesis*

All domains of computation are complete partial orders with a least element.

All computable functions are continuous.

Cpo's and domains

A chain complete poset, or cpo for short, is a poset $(D, \sqsubseteq)$ in which all countable increasing chains $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots$ have least upper bounds, $\bigsqcup_{n \geq 0} d_n$:

$$\forall m \geq 0 \,.\, d_m \sqsubseteq \bigsqcup_{n \geq 0} d_n \qquad \text{(lub1)}$$

$$\forall d \in D \,.\, (\forall m \geq 0 \,.\, d_m \sqsubseteq d) \;\Rightarrow\; \bigsqcup_{n \geq 0} d_n \sqsubseteq d. \qquad \text{(lub2)}$$

A domain is a cpo that possesses a least element, $\perp$:

$$\forall d \in D \,.\, \perp \sqsubseteq d.$$

Poset $D, \sqsubseteq$

$d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \cdots$

$(n \in \mathbb{N})$

complete

$\bigsqcup_{n \in \mathbb{N}} d_n$

lub = least upper bound

(1) It is an upper bound

$\forall i \in \mathbb{N}.\ d_i \sqsubseteq \bigsqcup_n d_n$

(2) Least

$\forall d \in D.\ \forall i \in \mathbb{N}.\ d_i \sqsubseteq d$

$\bigsqcup_n d_n \sqsubseteq d$

$$\frac{}{\bot \sqsubseteq x}$$

$$\frac{}{x_i \sqsubseteq \bigsqcup_{n \geq 0} x_n} \quad (i \geq 0 \text{ and } \langle x_n \rangle \text{ a chain})$$

$$\frac{\forall n \geq 0 \,.\, x_n \sqsubseteq x}{\bigsqcup_{n \geq 0} x_n \sqsubseteq x} \quad (\langle x_i \rangle \text{ a chain})$$

Domain of partial functions, $X \rightharpoonup Y$

Underlying set: all partial functions, f, with domain of definition $dom(f) \subseteq X$ and taking values in Y.

Partial order:

$$f \sqsubseteq g \quad \text{iff} \quad dom(f) \subseteq dom(g) \text{ and } \forall x \in dom(f).\ f(x) = g(x)$$

$$\text{iff} \quad graph(f) \subseteq graph(g)$$

Lub of chain $f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \dots$ is the partial function f with $dom(f) = \bigcup_{n \geq 0} dom(f_n)$ and

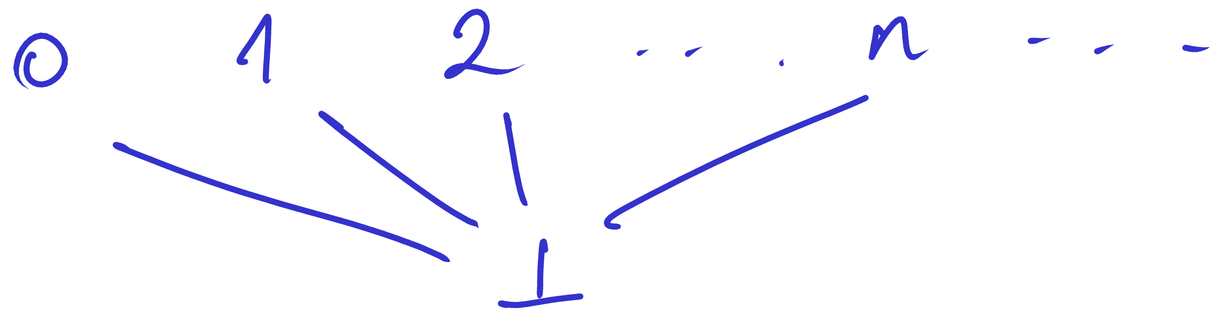
$$f(x) = \begin{cases} f_n(x) & \text{if } x \in dom(f_n), \text{ some } n \\ \text{undefined} & \text{otherwise} \end{cases}$$

Least element $\perp$ is the totally undefined partial function.

Domain of natural numbers.

($n \in \mathbb{N}$)

$\mathbb{N}_\perp =$



$f : \mathbb{N}_\perp \to \mathbb{N}_\perp$

computable $\Rightarrow$ monotone

$\Downarrow$

$\perp \sqsubseteq x \quad \forall x \in \mathbb{N}_\perp$

$f(\perp) \sqsubseteq f(x) \in \mathbb{N}_\perp$

$\quad \| $

$\quad 7$

$f(\perp) = 7 \in \mathbb{N}$

$\Downarrow$

f is the constantly 7 function

Every function

$$f : \mathbb{N}_\perp \to \mathbb{N}_\perp$$

s.t.

$$f(\perp) = \perp$$

is monotone.

Some properties of lubs of chains

Let D be a cpo.

1. For $d \in D$, $\bigsqcup_n d = d$.
2. For every chain $d_0 \sqsubseteq d_1 \sqsubseteq \ldots \sqsubseteq d_n \sqsubseteq \ldots$ in D,

$$\bigsqcup_n d_n = \bigsqcup_n d_{N+n}$$

   for all $N \in \mathbb{N}$.

$$\dfrac{\vdots \qquad\qquad \vdots}{\bigsqcup_n d_{N+n} \sqsubseteq \bigsqcup_n d_n}$$

$$\dfrac{\dfrac{\dfrac{}{d_i \sqsubseteq d_{N+i}} \qquad \dfrac{}{d_{N+i} \sqsubseteq \bigsqcup_n d_{N+n}}}{\forall i \qquad d_i \sqsubseteq \bigsqcup_n d_{N+n}}}{\bigsqcup_n d_n \sqsubseteq \bigsqcup_n d_{N+n}}$$

3. For every pair of chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots \sqsubseteq d_n \sqsubseteq \ldots$ and $e_0 \sqsubseteq e_1 \sqsubseteq \ldots \sqsubseteq e_n \sqsubseteq \ldots$ in D,

$$\text{if } d_n \sqsubseteq e_n \text{ for all } n \in \mathbb{N} \text{ then } \bigsqcup_n d_n \ \sqsubseteq\ \bigsqcup_n e_n.$$

$$\begin{array}{cccccccc} d_0 & \sqsubseteq & d_1 & \sqsubseteq \cdots & d_n & \sqsubseteq \cdots & \bigsqcup_n d_n \\ \sqcap\!\!\!\mid & & \sqcap\!\!\!\mid & & \sqcap\!\!\!\mid & & \sqcap\!\!\!\mid\ ? \\ e_0 & \sqsubseteq & e_1 & \sqsubseteq \cdots & e_n & \sqsubseteq \cdots & \bigsqcup_n e_n \end{array}$$

3. For every pair of chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots \sqsubseteq d_n \sqsubseteq \ldots$ and $e_0 \sqsubseteq e_1 \sqsubseteq \ldots \sqsubseteq e_n \sqsubseteq \ldots$ in D,

if $d_n \sqsubseteq e_n$ for all $n \in \mathbb{N}$ then $\bigsqcup_n d_n \ \sqsubseteq \ \bigsqcup_n e_n$.

$$\frac{\forall n \geq 0 \, . \, x_n \sqsubseteq y_n}{\bigsqcup_n x_n \sqsubseteq \bigsqcup_n y_n} \quad (\langle x_n \rangle \text{ and } \langle y_n \rangle \text{ chains})$$

Diagonalising a double chain

Lemma. *Let D be a cpo. Suppose that the doubly-indexed family of elements $d_{m,n} \in D$ ($m, n \geq 0$) satisfies*

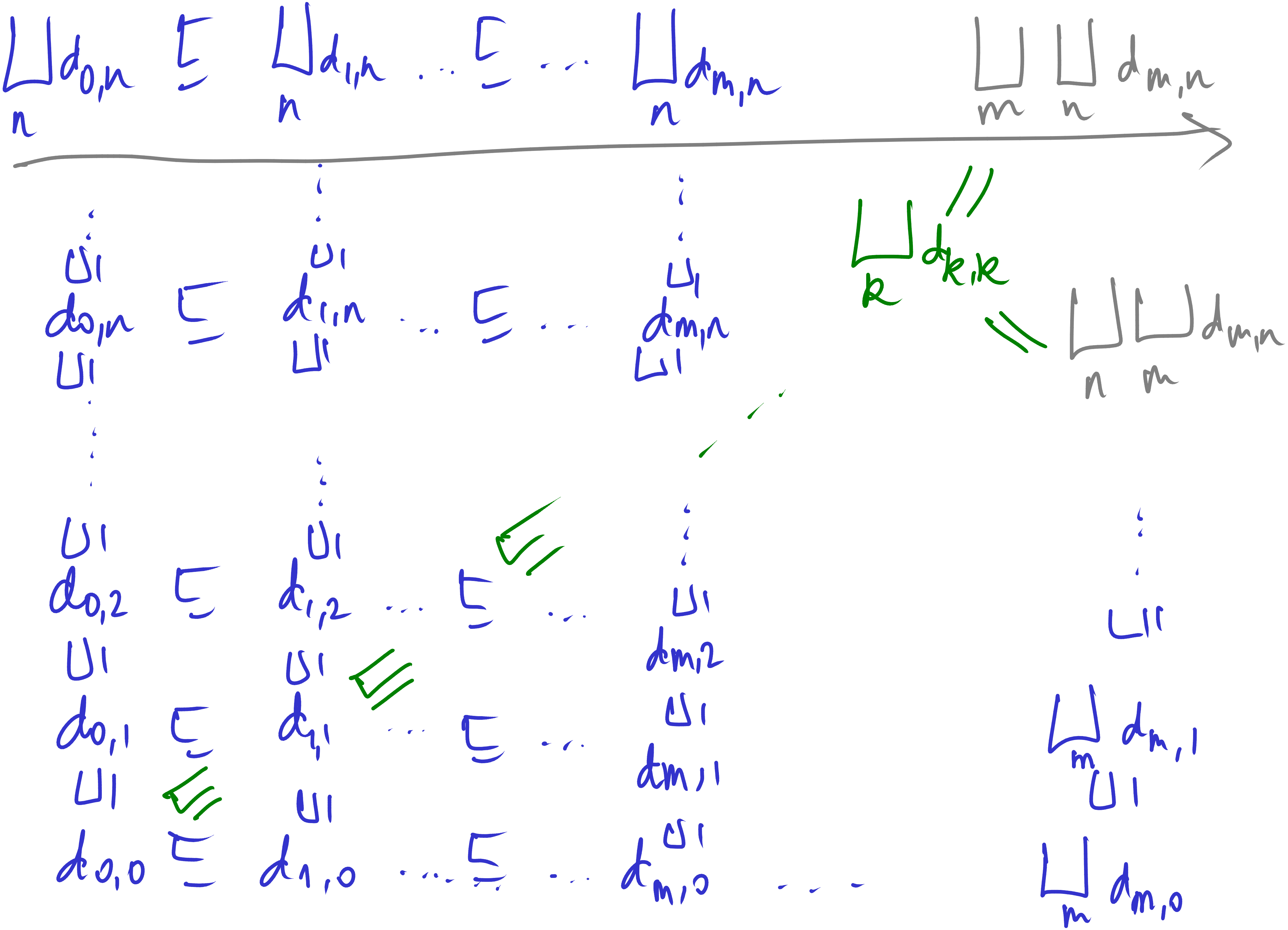
$$m \leq m' \mathbin{\&} n \leq n' \;\Rightarrow\; d_{m,n} \sqsubseteq d_{m',n'}. \qquad (\dagger)$$

Then

$$\bigsqcup_{n\geq 0} d_{0,n} \;\sqsubseteq\; \bigsqcup_{n\geq 0} d_{1,n} \;\sqsubseteq\; \bigsqcup_{n\geq 0} d_{2,n} \;\sqsubseteq\; \ldots$$

and

$$\bigsqcup_{m\geq 0} d_{m,0} \;\sqsubseteq\; \bigsqcup_{m\geq 0} d_{m,1} \;\sqsubseteq\; \bigsqcup_{m\geq 0} d_{m,3} \;\sqsubseteq\; \ldots$$



$\bigsqcup_n d_{0,n} \sqsubseteq \bigsqcup_n d_{1,n} \cdots \sqsubseteq \cdots \bigsqcup_n d_{m,n}$
$\bigsqcup_m \bigsqcup_n d_{m,n}$
$\bigsqcup_k d_{k,k}$
$\bigsqcup_n \bigsqcup_m d_{m,n}$
$d_{0,n} \sqsubseteq d_{1,n} \cdots \sqsubseteq \cdots d_{m,n}$
$d_{0,2} \sqsubseteq d_{1,2} \cdots \sqsubseteq \cdots d_{m,2}$
$d_{0,1} \sqsubseteq d_{1,1} \cdots \sqsubseteq \cdots d_{m,1}$
$d_{0,0} \sqsubseteq d_{1,0} \cdots \sqsubseteq \cdots d_{m,0} \cdots$
$\bigsqcup_m d_{m,1}$
$\bigsqcup_m d_{m,0}$

$$d_{m,n} \overset{\forall n}{\sqsubseteq} \bigsqcup_n d_{m,n} \overset{\forall m}{\sqsubseteq} \bigsqcup_m \left(\bigsqcup_n d_{m,n} \right)$$

$$\frac{d_{m,k} \sqsubseteq \bigsqcup_n d_{m,n}}{\bigsqcup_m d_{m,k} \sqsubseteq \bigsqcup_m \bigsqcup_n d_{m,n}}$$

$$d_{k,k} \sqsubseteq \bigsqcup_m d_{m,k}$$

$$\frac{\forall k.\quad d_{k,k} \sqsubseteq \bigsqcup_m \bigsqcup_n d_{m,n}}{\bigsqcup_k d_{k,k} \sqsubseteq \bigsqcup_m \bigsqcup_n d_{m,n}}$$

$$d_{k,k} \sqsubseteq \bigsqcup_k d_{k,k}$$

$$\begin{array}{c}
d_{m,n} \sqsubseteq d_{\max(m,n),\max(m,n)} \sqsubseteq \bigsqcup_k d_{k,k} \\
\hline
d_{m,n} \sqsubseteq \bigsqcup_k d_{k,k} \\
\hline
\bigsqcup_n d_{m,n} \sqsubseteq \bigsqcup_k d_{k,k} \\
\hline
\bigsqcup_m \bigsqcup_n d_{m,n} \sqsubseteq \bigsqcup_k d_{k,k}
\end{array}$$

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Then

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and

$$\bigsqcup_{m \geq 0} d_{m,0} \;\sqsubseteq\; \bigsqcup_{m \geq 0} d_{m,1} \;\sqsubseteq\; \bigsqcup_{m \geq 0} d_{m,3} \;\sqsubseteq\; \ldots$$

Moreover

$$\bigsqcup_{m \geq 0} \left(\bigsqcup_{n \geq 0} d_{m,n} \right) = \bigsqcup_{k \geq 0} d_{k,k} = \bigsqcup_{n \geq 0} \left(\bigsqcup_{m \geq 0} d_{m,n} \right) \;.$$

Continuity and strictness

- If D and E are cpo's, the function f is continuous iff
  1. it is monotone, and $\quad d \sqsubseteq d' \Rightarrow f(d) \sqsubseteq f(d')$.
  2. it preserves lubs of chains, *i.e.* for all chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots$ in D, it is the case that

$$\boxed{f(\bigsqcup_{n \geq 0} d_n) = \bigsqcup_{n \geq 0} f(d_n)} \text{ in } E.$$

$$D \xrightarrow{f} E$$

$$d_0 \sqsubseteq d_1 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \cdots \sqsubseteq \bigsqcup_n d_n$$

$$f(d_0) \sqsubseteq f(d_1) \sqsubseteq \cdots \sqsubseteq f(d_n) \sqsubseteq \cdots \sqsubseteq \bigsqcup_n f(d_n)$$

$$f(\bigsqcup_n d_n) \sqsupseteq \bigsqcup_n f(d_n)$$

Domain of streams (of natural numbers)

elements: infinite lists from $\mathbb{N}_\perp$, say $n_0, n_1, n_2, \ldots, n_k, \ldots$ $(k \in \mathbb{N})$

s.t. $n_i = \perp \implies n_j = \perp \;\; \forall j \geq i$

ex: $\perp \; \perp \; \perp \; \perp \; \perp \; \ldots \; \perp \; \ldots$

$1 \; \perp \; \perp \; \perp \; \perp \; \ldots \; \perp \; \ldots$

$1 \; 2 \; \perp \; \perp \; \ldots$

$$n_0\, n_1\, n_2 \dots n_k \dots \sqsubseteq m_0\, m_1\, m_2 \dots m_k \dots$$

$$\iff_{def} \quad n_i \sqsubseteq m_i \quad \forall i$$

$$0\ 1\ 2\ 3\ \bot\ \bot\ \bot\ \bot \sqsubseteq 0\ 1\ 2\ 3\ 4\ 5 \dots$$

▶ We have a domain!

$$\bigsqcup \left(\begin{array}{l} \bot\ \bot\ \bot \dots \bot \dots \\ \sqcap \\ 0\ \bot\ \bot \dots \bot \dots \\ \sqcap \\ 0\ 1\ \bot \dots \bot \dots \\ \sqcap \\ 0\ 1\ 2\ \bot \dots \bot \dots \\ \sqcap \\ 0\ 1 \dots n\ \bot\ \bot \dots \end{array} \right) = 0\,1\,2\,3 \dots n \dots$$

$$\underline{\mathrm{Stream}}(\mathbb{N}) \xrightarrow[\text{continuous}]{f} \mathbb{N}_\perp$$

Intuitively computable functions look at finite prefixes of the input a finite lot!

Exercise

$$f(n_0\, n_1\, n_2 \ldots n_k \ldots) = m \in \mathbb{N}$$

$\overset{\Longrightarrow}{\text{By continuity}}$ $\exists\, l \in \mathbb{N}.\ f(n_0\, n_1 \ldots n_l \perp \perp \perp \perp) = m$