Fixed point property of $\llbracket \mathbf{while} \ B \ \mathbf{do} \ C \rrbracket$

 $\llbracket \mathbf{while} \ B \ \mathbf{do} \ C \rrbracket = f_{\llbracket B \rrbracket, \llbracket C \rrbracket}(\llbracket \mathbf{while} \ B \ \mathbf{do} \ C \rrbracket)$ where, for each $b : State \to \{true, false\}$ and $c : State \to State$, we define

$$f_{b,c}: (State \rightarrow State) \rightarrow (State \rightarrow State)$$

as

$$f_{b,c} = \lambda w \in (State \rightharpoonup State). \ \lambda s \in State. \ if (b(s), w(c(s)), s).$$

- Why does $w = f_{\llbracket B \rrbracket, \llbracket C \rrbracket}(w)$ have a solution?
- What if it has several solutions—which one do we take to be
 [while B do C]?

Approximating [while $B \operatorname{do} C$] $S \operatorname{fate} \rightarrow S \operatorname{fate}$

Wo E [[nhile Bdo C]] Il def L'empty partial function FIBJ, NCM (Wo)(S) E Tuhle B do C] $= if(IBM(s), W_{0}(ICMs), s)$ $= \operatorname{FE}(\operatorname{FB}(s), \uparrow, s)$

Wntig füßster (Wn)

 $\omega_2 = fasy, acy(\omega)$ $\omega_2(s) = \mathcal{I}(IB]s, \omega_1(ICIs), s)$ $= j(TBJS, f(TBJ(TCJS), \uparrow, TCJS),$ (ک

Wn Ellwhile 3 dr CM

 $\omega_0 \leq \omega_1 \leq \cdots \leq \omega_n \leq \omega_{nn} \leq \cdots$

 $\bigcup_{n} w_{n} = [[u_{n}]_{i} le B do C]$ Join The union of the graphs of the wr's.

 $\frac{?}{f_{\pi}} \int f_{\pi} \mathcal{D}_{\pi} \mathcal{D}_{\pi} \left(\bigcup_{n} \mathcal{U}_{n} \right) = \bigcup_{n} \mathcal{U}_{n} \qquad ?$

Approximating $\llbracket while B \operatorname{do} C \rrbracket$

$$\begin{split} f_{\llbracket B \rrbracket, \llbracket C \rrbracket}^{n}(\bot) \\ &= \lambda s \in State. \\ & \left\{ \begin{array}{ll} \llbracket C \rrbracket^{k}(s) & \text{if } \exists \ 0 \leq k < n. \ \llbracket B \rrbracket(\llbracket C \rrbracket^{k}(s)) = false \\ & \text{and } \forall \ 0 \leq i < k. \ \llbracket B \rrbracket(\llbracket C \rrbracket^{i}(s)) = true \\ \uparrow & \text{if } \forall \ 0 \leq i < n. \ \llbracket B \rrbracket(\llbracket C \rrbracket^{i}(s)) = true \end{array} \right. \end{split}$$

$$D \stackrel{\mathrm{def}}{=} (State \rightharpoonup State)$$

• Partial order \sqsubseteq on D:

 $w \sqsubseteq w'$ iff for all $s \in State$, if w is defined at s then so is w' and moreover w(s) = w'(s).

iff the graph of w is included in the graph of w'.

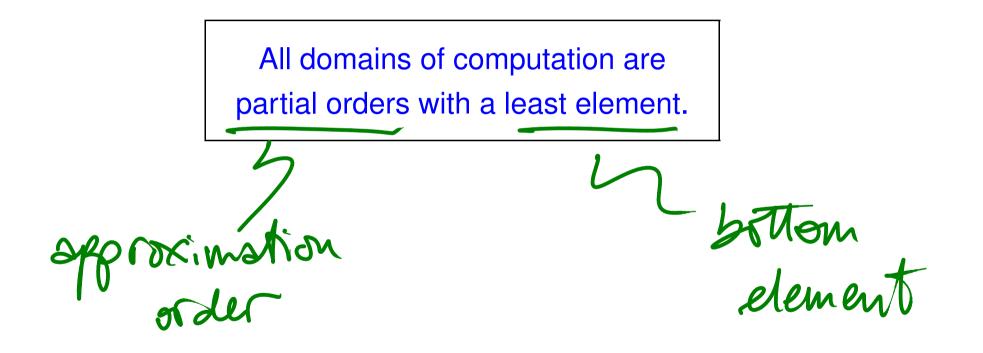
- Least element $\perp \in D$ w.r.t. \sqsubseteq :
 - \perp = totally undefined partial function
 - = partial function with empty graph

(satisfies $\perp \sqsubseteq w$, for all $w \in D$).

Topic 2

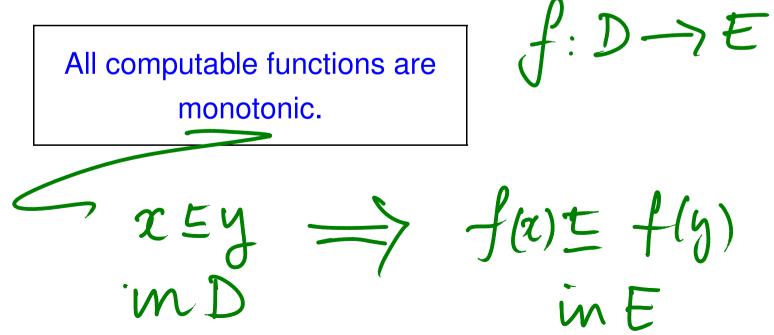
Least Fixed Points

Thesis



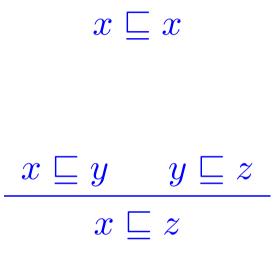
Thesis

All domains of computation are partial orders with a least element.



Partially ordered sets

A binary relation \sqsubseteq on a set D is a partial order iff it is reflexive: $\forall d \in D. \ d \sqsubseteq d$ transitive: $\forall d, d', d'' \in D. \ d \sqsubseteq d' \sqsubseteq d'' \Rightarrow d \sqsubseteq d''$ anti-symmetric: $\forall d, d' \in D. \ d \sqsubseteq d' \sqsubseteq d \Rightarrow d = d'.$ Such a pair (D, \sqsubseteq) is called a partially ordered set, or poset.



$$\frac{x \sqsubseteq y \quad y \sqsubseteq x}{x = y}$$

Domain of partial functions, $X \rightharpoonup Y$

 $f \subseteq g: X \rightarrow Y$ $\eta = \int f graph(f) \subseteq graph(g).$ $\int = \phi : X \rightarrow Y.$

Domain of partial functions, $X \rightharpoonup Y$

Underlying set: all partial functions, f, with domain of definition $dom(f) \subseteq X$ and taking values in Y.

Partial order:

$$\begin{array}{ll} f\sqsubseteq g & \text{iff} & dom(f)\subseteq dom(g) \text{ and} \\ & \forall x\in dom(f). \ f(x)=g(x) \\ & \text{iff} & graph(f)\subseteq graph(g) \end{array}$$

• A function $f: D \to E$ between posets is monotone iff $\forall d, d' \in D. \ d \sqsubseteq d' \Rightarrow f(d) \sqsubseteq f(d').$

$$\frac{x \sqsubseteq y}{f(x) \sqsubseteq f(y)} \quad (f \text{ monotone})$$

frog, rcg: (81ste ~ State) ~ (81ste ~ State) $\begin{array}{l} & \begin{array}{c} & \end{array} \\ & \begin{array}{c} \lambda & \end{array} \end{array} \\ & \begin{array}{c} \lambda & \end{array} \end{array} \\ & \begin{array}{c} \lambda & \end{array} \\ & \begin{array}{c} \lambda & \end{array} \\ & \begin{array}{c} \lambda & \end{array} \end{array} \\ & \begin{array}{c} \lambda & \end{array} \\ & \begin{array}{c} \lambda & \end{array} \end{array} \\ \\ \\ & \begin{array}{c} \lambda & \end{array} \end{array} \\ \\ \\ & \begin{array}{c} \lambda & \end{array} \end{array} \\ \\ \\ \end{array} \\ \\ \end{array} \\ \\ \end{array} \\ \\ \end{array} \\ \\ \begin{array}{c} \lambda & \end{array} \end{array} \\ \\ \end{array} \\ \begin{array}{c} \lambda & \end{array} \\ \\ \\ \end{array} \\ \\ \\ \\ \end{array} \\ \\ \end{array} \\ \\ \\ \end{array} \\ \\ \end{array} \\ \\ \\ \end{array} \\ \\ \\ \end{array} \\ \\ \\ \end{array} \\ \\ \end{array} \\ \\ \\ \\ \end{array} \\ \\ \\ \end{array} \\ \\ \\ \end{array} \\ \\ \end{array} \\ \\ \\ \\ \end{array} \\ \\ \end{array} \\ \\ \end{array} \\ \\ \end{array} \\ \\ \\ \\$

 $f_{\overline{i}} \delta \mathcal{Y}, f_{\overline{i}} c \mathcal{Y}(\omega)(s) \downarrow$ $-q(\pi BMG), \omega(\pi CM(S)), S)$ い生い (i) [TBN(s)= drue $\implies f \overline{\eta} \overline{\beta} \overline{\eta}, \overline{\kappa} \overline{\gamma} (\omega)(s) = \omega(\overline{\eta} c \eta s)$ $\Rightarrow \omega(\overline{TCT})s) J and \omega(\overline{TCT}s) = \omega(\overline{TCT}s)$ \rightarrow fibration(w')(s) 1

Least Elements

Suppose that D is a poset and that S is a subset of D.

An element $d \in S$ is the *least* element of S if it satisfies

$$\forall x \in S. d \sqsubseteq x .$$

$$d_0 \text{ is the least element of } S. \Longrightarrow d_0 \subseteq d_1 = d_0 d_1 \\ d_1 \text{ is the least element of } S. \Longrightarrow d_n \subseteq d_0 = d_0 d_1 d_0 d_1$$

- Note that because is anti-symmetric, S has at most one least element.
- Note also that a poset may not have least element.

fix(f)

Let D be a poset and $f: D \to D$ be a function. An element $d \in D$ is a pre-fixed point of f if it satisfies $f(d) \sqsubseteq d$.

The *least pre-fixed point* of f, if it exists, will be written Meningful

It is thus (uniquely) specified by the two properties:

$$\begin{split} f(fix(f)) &\sqsubseteq fix(f) & \text{(Ifp1)} \\ \forall d \in D. \ f(d) &\sqsubseteq d \ \Rightarrow \ fix(f) &\sqsubseteq d. & \text{(Ifp2)} \end{split}$$

1.

 $f(fix(f)) \sqsubseteq fix(f)$

2. Let D be a poset and let $f : D \to D$ be a function with a least pre-fixed point $fix(f) \in D$. For all $x \in D$, to prove that $fix(f) \sqsubseteq x$ it is enough to establish that $f(x) \sqsubseteq x$.

$$\frac{f(x) \sqsubseteq x}{fix(f) \sqsubseteq x}$$

f: D > D monotone D poet $fix(f) \implies f(fx(f)) = fx(f)$

Least pre-fixed points are fixed points

If it exists, the least pre-fixed point of a mononote function on a partial order is necessarily a fixed point.

fa)5x fxf,5x $f(fact) \subseteq fox(f)$ (f mon) f(ffref) 5 f(fref) $fix f \in f(fx f)$ f(finf) = fia(f) $f(f_{\alpha}(f_{1})) = f_{\alpha}(f_{1})$

Thesis*

All domains of computation are

complete partial orders with a least element.

All computable functions are continuous.