Denotational Semantics

Lectures for Part II CST 2021/22

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Course web page:
http://www.cl.cam.ac.uk/teaching/2122/DenotSem/
Topic 1

Introduction
What is this course about?

- General area.
  
  *Formal methods*: Mathematical techniques for the specification, development, and verification of software and hardware systems.

- Specific area.
  
  *Formal semantics*: Mathematical theories for ascribing meanings to computer languages.
Why do we care?
Why do we care?

- Rigour.
  - ... specification of programming languages
  - ... justification of program transformations
Why do we care?

- Rigour.
  - specification of programming languages
  - justification of program transformations

- Insight.
  - generalisations of notions computability
  - higher-order functions
  - data structures
Feedback into language design.

... continuations
... monads
● Feedback into language design.
    ... continuations
    ... monads

● Reasoning principles.
    ... Scott induction
    ... Logical relations
    ... Co-induction
Styles of formal semantics

Operational.

Axiomatic.

Denotational.
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Meanings for program phrases defined in terms of the steps of computation they can take during program execution.

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Meanings for program phrases defined in terms of the *steps of computation* they can take during program execution.

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Meanings for program phrases defined indirectly via the *axioms and rules* of some logic of program properties.

Denotational.
Styles of formal semantics

Operational.
Meanings for program phrases defined in terms of the *steps of computation* they can take during program execution.

Axiomatic.
Meanings for program phrases defined indirectly via the *axioms and rules* of some logic of program properties.

Denotational.
Concerned with giving *mathematical models* of programming languages. Meanings for program phrases defined abstractly as elements of some suitable mathematical structure.
Basic idea of denotational semantics

Syntax \[\mapsto\] Semantics

\[ P \mapsto [P] \]
Basic idea of denotational semantics

Syntax $\rightarrow$ Semantics

Recursive program $\leftrightarrow$ Partial recursive function

$P \leftrightarrow [P]$
Basic idea of denotational semantics

Syntax $\xrightarrow{\ldots}$ Semantics

Recursive program $\mapsto$ Partial recursive function

Boolean circuit $\mapsto$ Boolean function

$P$ $\mapsto$ $[P]$
Basic idea of denotational semantics

Syntax $\rightarrow$ Semantics

- Recursive program $\mapsto$ Partial recursive function
- Boolean circuit $\mapsto$ Boolean function

$P \mapsto [P]

Concerns:

- Abstract models (i.e. implementation/machine independent).
  \[\sim\] Lectures 2, 3 and 4.
Basic idea of denotational semantics

Syntax $\overset{\rightarrow}{\longrightarrow}$ Semantics

Recursive program $\mapsto$ Partial recursive function

Boolean circuit $\mapsto$ Boolean function

$P \mapsto [P]$ 

Concerns:

- Abstract models (i.e. implementation/machine independent).
  $\leadsto$ Lectures 2, 3 and 4.

- Compositionality.
  $\leadsto$ Lectures 5 and 6.
Basic idea of denotational semantics

Syntax $\xrightarrow{\square}$ Semantics

Recursive program $\mapsto$ Partial recursive function
Boolean circuit $\mapsto$ Boolean function

$P$ $\mapsto$ $[P]$  

Concerns:

- Abstract models (i.e. implementation/machine independent).
  $\rightsquigarrow$ Lectures 2, 3 and 4.

- Compositionality.
  $\rightsquigarrow$ Lectures 5 and 6.

- Relationship to computation (e.g. operational semantics).
  $\rightsquigarrow$ Lectures 7 and 8.
Characteristic features of a denotational semantics

- Each phrase (= part of a program), \( P \), is given a denotation, \( [P] \) — a mathematical object representing the contribution of \( P \) to the meaning of any complete program in which it occurs.

- The denotation of a phrase is determined just by the denotations of its subphrases (one says that the semantics is compositional).
Basic example of denotational semantics (I)

IMP$^-$ syntax

Arithmetic expressions

\[ A \in A_{\text{exp}} ::=} \ n \mid L \mid A + A \mid \ldots \]
where \( n \) ranges over integers and \( L \) over a specified set of locations \( L \)

Boolean expressions

\[ B \in B_{\text{exp}} ::=} \ true \mid false \mid A = A \mid \ldots \]
\[ \mid \neg B \mid \ldots \]

Commands

\[ C \in \text{Comm} ::=} \ skip \mid L := A \mid C; C \]
\[ \mid \text{if } B \text{ then } C \text{ else } C \]
Basic example of denotational semantics (II)

Semantic functions

\( A : \ \text{Aexp} \rightarrow (\text{State} \rightarrow \mathbb{Z}) \)

where

\[ \mathbb{Z} = \{ \ldots, -1, 0, 1, \ldots \} \]

\[ \text{State} = (\mathbb{L} \rightarrow \mathbb{Z}) \]
Basic example of denotational semantics (II)

Semantic functions

\[ A : \ Aexp \rightarrow (State \rightarrow \mathbb{Z}) \]
\[ B : \ Bexp \rightarrow (State \rightarrow \mathbb{B}) \]

where

\[ \mathbb{Z} = \{ \ldots, -1, 0, 1, \ldots \} \]
\[ \mathbb{B} = \{ true, false \} \]
\[ State = (\mathbb{L} \rightarrow \mathbb{Z}) \]
Basic example of denotational semantics (II)

Semantic functions

\[ A : \text{Aexp} \rightarrow (\text{State} \rightarrow \mathbb{Z}) \]
\[ B : \text{Bexp} \rightarrow (\text{State} \rightarrow \mathbb{B}) \]
\[ C : \text{Comm} \rightarrow (\text{State} \rightarrow \text{State}) \]

where

\[ \mathbb{Z} = \{ \ldots, -1, 0, 1, \ldots \} \]
\[ \mathbb{B} = \{ \text{true}, \text{false} \} \]
\[ \text{State} = (\mathbb{L} \rightarrow \mathbb{Z}) \]
Basic example of denotational semantics (III)

Semantic function $\mathcal{A}$

\[
\mathcal{A}[n] = \lambda s \in \text{State}. n
\]

\[
\mathcal{A}[L] = \lambda s \in \text{State}. s(L)
\]

\[
\mathcal{A}[A_1 + A_2] = \lambda s \in \text{State}. \mathcal{A}[A_1](s) + \mathcal{A}[A_2](s)
\]
Basic example of denotational semantics (IV)

Semantic function $\mathcal{B}$

\[ \mathcal{B}[true] = \lambda s \in State. \text{true} \]
\[ \mathcal{B}[false] = \lambda s \in State. \text{false} \]
\[ \mathcal{B}[A_1 = A_2] = \lambda s \in State. \text{eq}(\mathcal{A}[A_1](s), \mathcal{A}[A_2](s)) \]

where $\text{eq}(a, a') = \begin{cases} 
\text{true} & \text{if } a = a' \\
\text{false} & \text{if } a \neq a' 
\end{cases}$
Basic example of denotational semantics (V)

Semantic function $C$

$$[[\text{skip}]] = \lambda s \in \text{State}. s$$

**NB:** From now on the names of semantic functions are omitted!
A simple example of compositionality

Given partial functions $[C], [C'] : \text{State} \rightarrow \text{State}$ and a function $[B] : \text{State} \rightarrow \{\text{true}, \text{false}\}$, we can define

$$\text{[if } B \text{ then } C \text{ else } C'\text{]} = \lambda s \in \text{State. if} ([B](s), [C](s), [C'])(s)$$

where

$$\text{if} (b, x, x') = \begin{cases} x & \text{if } b = \text{true} \\ x' & \text{if } b = \text{false} \end{cases}$$
Basic example of denotational semantics (VI)

Semantic function $C$

$$[L := A] = \lambda s \in \text{State}. \lambda \ell \in \mathbb{L}. \text{if } (\ell = L, [A](s), s(\ell))$$
Denotational semantics of sequential composition

Denotation of sequential composition $C; C'$ of two commands

$$\left[ C; C' \right] = \left[ C' \right] \circ \left[ C \right] = \lambda s \in State. \left[ C' \right] \left( \left[ C \right] (s) \right)$$

given by composition of the partial functions from states to states $\left[ C \right], \left[ C' \right] : State \rightarrow State$ which are the denotations of the commands.
Denotational semantics of sequential composition

Denotation of sequential composition \( C; C' \) of two commands

\[
[C; C'] = [C'] \circ [C] = \lambda s \in \text{State}. \, [C']([C](s))
\]

given by composition of the partial functions from states to states \([C], [C'] : \text{State} \rightarrow \text{State}\) which are the denotations of the commands.

Cf. operational semantics of sequential composition:

\[
\begin{align*}
C, s & \Downarrow s' & C', s' & \Downarrow s'' & \quad \Rightarrow \\
C; C', s & \Downarrow s''
\end{align*}
\]
\[ \text{while } B \text{ do } C \]
Fixed point property of 

\[ [\text{while } B \text{ do } C] \]

\[ [\text{while } B \text{ do } C] = f_{[B],[C]}([\text{while } B \text{ do } C]) \]

where, for each \( b : \text{State} \rightarrow \{\text{true, false}\} \) and 
\( c : \text{State} \rightarrow \text{State} \), we define 

\[ f_{b,c} : (\text{State} \rightarrow \text{State}) \rightarrow (\text{State} \rightarrow \text{State}) \]

as 

\[ f_{b,c} = \lambda w \in (\text{State} \rightarrow \text{State}). \lambda s \in \text{State}. \text{if} (b(s), w(c(s)), s) \].
Fixed point property of

\[ [\text{while } B \text{ do } C] \]

\[ [\text{while } B \text{ do } C] = f_{[B],[C]}([\text{while } B \text{ do } C]) \]

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as

\[ f_{b,c} = \lambda w \in (\text{State} \rightarrow \text{State}). \lambda s \in \text{State}. \text{if}(b(s), w(c(s)), s). \]

- Why does \( w = f_{[B],[C]}(w) \) have a solution?
- What if it has several solutions—which one do we take to be \([\text{while } B \text{ do } C]\)?
Approximating \( \text{while } B \text{ do } C \)
Approximating $[\text{while } B \text{ do } C]$ 

\[
f_{[B],[C]}^n(\bot) = \lambda s \in \text{State.} \begin{cases} 
[C]^k(s) & \text{if } \exists 0 \leq k < n. [B]([C]^k(s)) = \text{false} \\
& \text{and } \forall 0 \leq i < k. [B]([C]^i(s)) = \text{true} \\
\uparrow & \text{if } \forall 0 \leq i < n. [B]([C]^i(s)) = \text{true}
\end{cases}
\]
\[ D \overset{\text{def}}{=} (\text{State} \rightarrow \text{State}) \]

- **Partial order** \( \sqsubseteq \) on \( D \):
  \[ w \sqsubseteq w' \quad \text{iff} \quad \text{for all } s \in \text{State}, \text{ if } w \text{ is defined at } s \text{ then so is } w' \text{ and moreover } w(s) = w'(s). \]
  \[ \text{iff} \quad \text{the graph of } w \text{ is included in the graph of } w'. \]

- **Least element** \( \bot \in D \) w.r.t. \( \sqsubseteq \):
  \[ \bot = \text{totally undefined partial function} = \text{partial function with empty graph} \]
  (satisfies \( \bot \sqsubseteq w \), for all \( w \in D \)).
Topic 2

Least Fixed Points
Thesis

All domains of computation are partial orders with a least element.
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All domains of computation are partial orders with a least element.

All computable functions are monotonic.
Partially ordered sets

A binary relation $\sqsubseteq$ on a set $D$ is a partial order iff it is

**reflexive:** $\forall d \in D. \ d \sqsubseteq d$

**transitive:** $\forall d, d', d'' \in D. \ d \sqsubseteq d' \sqsubseteq d'' \Rightarrow d \sqsubseteq d''$

**anti-symmetric:** $\forall d, d' \in D. \ d \sqsubseteq d' \sqsubseteq d \Rightarrow d = d'$.

Such a pair $(D, \sqsubseteq)$ is called a partially ordered set, or poset.
\[ x \sqsubseteq x \]

\[ x \sqsubseteq y \quad y \sqsubseteq z \]
\[ \quad \frac{}{x \sqsubseteq z} \]

\[ x \sqsubseteq y \quad y \sqsubseteq x \]
\[ \quad \frac{}{x = y} \]
Domain of partial functions, $X \rightarrow Y$
Domain of partial functions, $X \twoheadrightarrow Y$

**Underlying set:** all partial functions, $f$, with domain of definition $\text{dom}(f) \subseteq X$ and taking values in $Y$. 
Domain of partial functions, $X \rightarrow Y$

**Underlying set:** all partial functions, $f$, with domain of definition $\text{dom}(f) \subseteq X$ and taking values in $Y$.

**Partial order:**

$f \sqsubseteq g$ iff $\text{dom}(f) \subseteq \text{dom}(g)$ and $\forall x \in \text{dom}(f). f(x) = g(x)$

iff $\text{graph}(f) \subseteq \text{graph}(g)$
A function $f : D \rightarrow E$ between posets is monotone iff

$$\forall d, d' \in D. \ d \sqsubseteq d' \Rightarrow f(d) \sqsubseteq f(d').$$

$$x \sqsubseteq y \quad \frac{\quad}{f(x) \sqsubseteq f(y)} \quad (f \text{ monotone})$$
Least Elements

Suppose that $D$ is a poset and that $S$ is a subset of $D$.

An element $d \in S$ is the least element of $S$ if it satisfies

$$\forall x \in S. \ d \sqsubseteq x .$$

- Note that because $\sqsubseteq$ is anti-symmetric, $S$ has at most one least element.
- Note also that a poset may not have least element.
Pre-fixed points

Let $D$ be a poset and $f: D \to D$ be a function.

An element $d \in D$ is a pre-fixed point of $f$ if it satisfies $f(d) \sqsubseteq d$.

The least pre-fixed point of $f$, if it exists, will be written $\text{fix}(f)$.

It is thus (uniquely) specified by the two properties:

1. $f(\text{fix}(f)) \sqsubseteq \text{fix}(f)$ \hspace{1cm} (lfp1)
2. $\forall d \in D. f(d) \sqsubseteq d \Rightarrow \text{fix}(f) \sqsubseteq d$. \hspace{1cm} (lfp2)
2. Let $D$ be a poset and let $f : D \to D$ be a function with a least pre-fixed point $\text{fix}(f) \in D$.

For all $x \in D$, to prove that $\text{fix}(f) \sqsubseteq x$ it is enough to establish that $f(x) \sqsubseteq x$. 
2. Let $D$ be a poset and let $f : D \to D$ be a function with a least pre-fixed point $\text{fix}(f) \in D$.

For all $x \in D$, to prove that $\text{fix}(f) \sqsubseteq x$ it is enough to establish that $f(x) \sqsubseteq x$.

\[
\begin{align*}
  f(x) \sqsubseteq x \\
  \hline \\
  \text{fix}(f) \sqsubseteq x
\end{align*}
\]
Proof principle

1. 

\[ f(\text{fix}(f)) \subseteq \text{fix}(f) \]

2. Let \( D \) be a poset and let \( f : D \to D \) be a function with a least pre-fixed point \( \text{fix}(f) \in D \).

For all \( x \in D \), to prove that \( \text{fix}(f) \subseteq x \) it is enough to establish that \( f(x) \subseteq x \).

\[ f(x) \subseteq x \]

\[ \text{fix}(f) \subseteq x \]
Least pre-fixed points are fixed points

If it exists, the least pre-fixed point of a mononote function on a partial order is necessarily a fixed point.
All domains of computation are complete partial orders with a least element.
All domains of computation are complete partial orders with a least element.

All computable functions are continuous.
Cpo’s and domains

A chain complete poset, or cpo for short, is a poset \((D, \sqsubseteq)\) in which all countable increasing chains \(d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots\) have least upper bounds, \(\bigsqcup_{n \geq 0} d_n:\)

\[
\forall m \geq 0 . \ d_m \sqsubseteq \bigsqcup_{n \geq 0} d_n \tag{\text{lub1}}
\]

\[
\forall d \in D . \ (\forall m \geq 0 . \ d_m \sqsubseteq d) \ \Rightarrow \ \bigsqcup_{n \geq 0} d_n \sqsubseteq d. \tag{\text{lub2}}
\]

A domain is a cpo that possesses a least element, \(\bot:\)

\[
\forall d \in D . \ \bot \sqsubseteq d.
\]
∀n ≥ 0. x_n ⊑ x \quad \text{(} x_i \text{ a chain)}

\bigcup_{n≥0} x_n ⊑ x

(\langle x_n \rangle \text{ a chain})
Domain of partial functions, $X \rightarrow Y$
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**Underlying set**: all partial functions, $f$, with domain of definition $\text{dom}(f) \subseteq X$ and taking values in $Y$. 
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**Underlying set:** all partial functions, $f$, with domain of definition $\text{dom}(f) \subseteq X$ and taking values in $Y$.

**Partial order:**

$$f \sqsubseteq g \iff \text{dom}(f) \subseteq \text{dom}(g) \text{ and } \forall x \in \text{dom}(f). \ f(x) = g(x)$$

$$\iff \text{graph}(f) \subseteq \text{graph}(g)$$
Domain of partial functions, $X \rightarrow Y$

**Underlying set:** all partial functions, $f$, with domain of definition $\text{dom}(f) \subseteq X$ and taking values in $Y$.

**Partial order:**

\[ f \sqsubseteq g \text{ iff } \text{dom}(f) \subseteq \text{dom}(g) \text{ and } \forall x \in \text{dom}(f). f(x) = g(x) \]

\[ \text{iff } \text{graph}(f) \subseteq \text{graph}(g) \]

**Lub of chain** $f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \ldots$ is the partial function $f$ with $\text{dom}(f) = \bigcup_{n \geq 0} \text{dom}(f_n)$ and

\[
 f(x) = \begin{cases} 
 f_n(x) & \text{if } x \in \text{dom}(f_n), \text{ some } n \\
 \text{undefined} & \text{otherwise}
\end{cases}
\]
Domain of partial functions, \( X \rightarrow Y \)

**Underlying set:** all partial functions, \( f \), with domain of definition \( \text{dom}(f) \subseteq X \) and taking values in \( Y \).

**Partial order:**

\[ f \sqsubseteq g \quad \text{iff} \quad \text{dom}(f) \subseteq \text{dom}(g) \quad \text{and} \]

\[ \forall x \in \text{dom}(f). \ f(x) = g(x) \]

\[ \text{iff} \quad \text{graph}(f) \subseteq \text{graph}(g) \]

**Lub of chain** \( f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \ldots \) is the partial function \( f \) with

\[ \text{dom}(f) = \bigcup_{n \geq 0} \text{dom}(f_n) \] and

\[ f(x) = \begin{cases} f_n(x) & \text{if } x \in \text{dom}(f_n), \text{ some } n \\ \text{undefined} & \text{otherwise} \end{cases} \]

**Least element** \( \perp \) is the totally undefined partial function.
Some properties of lubs of chains

Let $D$ be a cpo.

1. For $d \in D$, $\bigsqcup_n d = d$.

2. For every chain $d_0 \sqsubseteq d_1 \sqsubseteq \ldots \sqsubseteq d_n \sqsubseteq \ldots$ in $D$,

$$\bigsqcup_n d_n = \bigsqcup_n d_{N+n}$$

for all $N \in \mathbb{N}$. 
3. For every pair of chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots \sqsubseteq d_n \sqsubseteq \ldots$ and $e_0 \sqsubseteq e_1 \sqsubseteq \ldots \sqsubseteq e_n \sqsubseteq \ldots$ in $D$, if $d_n \sqsubseteq e_n$ for all $n \in \mathbb{N}$ then $\bigcup_n d_n \sqsubseteq \bigcup_n e_n$. 
3. For every pair of chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots \sqsubseteq d_n \sqsubseteq \ldots$ and $e_0 \sqsubseteq e_1 \sqsubseteq \ldots \sqsubseteq e_n \sqsubseteq \ldots$ in $D$,

if $d_n \sqsubseteq e_n$ for all $n \in \mathbb{N}$ then $\bigsqcup_n d_n \sqsubseteq \bigsqcup_n e_n$.

$$\forall n \geq 0. x_n \sqsubseteq y_n \quad (\langle x_n \rangle \text{ and } \langle y_n \rangle \text{ chains})$$

$$\bigsqcup_n x_n \sqsubseteq \bigsqcup_n y_n$$
Lemma. Let $D$ be a cpo. Suppose that the doubly-indexed family of elements $d_{m,n} \in D$ ($m, n \geq 0$) satisfies
\[ m \leq m' \land n \leq n' \Rightarrow d_{m,n} \sqsubseteq d_{m',n'} . \] (†)

Then
\[ \bigsqcup_{n \geq 0} d_{0,n} \sqsubseteq \bigsqcup_{n \geq 0} d_{1,n} \sqsubseteq \bigsqcup_{n \geq 0} d_{2,n} \sqsubseteq \ldots \]

and
\[ \bigsqcup_{m \geq 0} d_{m,0} \sqsubseteq \bigsqcup_{m \geq 0} d_{m,1} \sqsubseteq \bigsqcup_{m \geq 0} d_{m,3} \sqsubseteq \ldots \]
Diagonalising a double chain

Lemma. Let $D$ be a cpo. Suppose that the doubly-indexed family of elements $d_{m,n} \in D$ ($m, n \geq 0$) satisfies

\[ m \leq m' \& n \leq n' \Rightarrow d_{m,n} \sqsubseteq d_{m',n'} . \]  

Then

\[ \bigsqcup_{n \geq 0} d_{0,n} \sqsubseteq \bigsqcup_{n \geq 0} d_{1,n} \sqsubseteq \bigsqcup_{n \geq 0} d_{2,n} \sqsubseteq \ldots \]

and

\[ \bigsqcup_{m \geq 0} d_{m,0} \sqsubseteq \bigsqcup_{m \geq 0} d_{m,1} \sqsubseteq \bigsqcup_{m \geq 0} d_{m,3} \sqsubseteq \ldots \]

Moreover

\[ \bigsqcup_{m \geq 0} \left( \bigsqcup_{n \geq 0} d_{m,n} \right) = \bigsqcup_{k \geq 0} d_{k,k} = \bigsqcup_{n \geq 0} \left( \bigsqcup_{m \geq 0} d_{m,n} \right) . \]
Continuity and strictness

- If $D$ and $E$ are cpo's, the function $f$ is continuous iff
  1. it is monotone, and
  2. it preserves lubs of chains, i.e. for all chains
     \[ d_0 \sqsubseteq d_1 \sqsubseteq \ldots \] in $D$, it is the case that
     \[ f(\bigsqcup_{n \geq 0} d_n) = \bigsqcup_{n \geq 0} f(d_n) \] in $E$. 

Continuity and strictness

- If $D$ and $E$ are cpo’s, the function $f$ is continuous iff
  1. it is monotone, and
  2. it preserves lubs of chains, i.e. for all chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots$ in $D$, it is the case that
     \[
     f(\bigsqcup_{n \geq 0} d_n) = \bigsqcup_{n \geq 0} f(d_n) \text{ in } E.
     \]

- If $D$ and $E$ have least elements, then the function $f$ is strict iff $f(\bot) = \bot$. 

Tarski’s Fixed Point Theorem

Let $f : D \rightarrow D$ be a continuous function on a domain $D$. Then

- $f$ possesses a least pre-fixed point, given by

$$\text{fix}(f) = \bigsqcup_{n \geq 0} f^n(\bot).$$

- Moreover, $\text{fix}(f)$ is a fixed point of $f$, i.e. satisfies

$$f(\text{fix}(f)) = \text{fix}(f),$$

and hence is the least fixed point of $f$. 
\[ \text{[while } B \text{ do } C]\]

\[ \text{[while } B \text{ do } C]\]

\[ = \text{fix}(f_{[B],[C]}) \]

\[ = \bigcup_{n \geq 0} f_{[B],[C]}^n(\bot) \]

\[ = \lambda s \in \text{State.} \quad \begin{cases} 
[C]^k(s) & \text{if } k \geq 0 \text{ is such that } [B](C)^k(s) = \text{false} \\
& \text{and } [B](C)^i(s) = \text{true} \text{ for all } 0 \leq i < k \\
\text{undefined} & \text{if } [B](C)^i(s) = \text{true} \text{ for all } i \geq 0 
\end{cases} \]
Topic 3

Constructions on Domains
For any set $X$, the relation of equality

\[ x \sqsubseteq x' \iff x = x' \quad (x, x' \in X) \]

makes $(X, \sqsubseteq)$ into a cpo, called the \textit{discrete} cpo with underlying set $X$. 

\[ x \sqsubseteq x' \iff x = x' \quad (x, x' \in X) \]
Discrete cpo’s and flat domains

For any set $X$, the relation of equality

$$x \sqsubseteq x' \iff x = x' \quad (x, x' \in X)$$

makes $(X, \sqsubseteq)$ into a cpo, called the discrete cpo with underlying set $X$.

Let $X_\bot \overset{\text{def}}{=} X \cup \{\bot\}$, where $\bot$ is some element not in $X$. Then

$$d \sqsubseteq d' \iff (d = d') \lor (d = \bot) \quad (d, d' \in X_\bot)$$

makes $(X_\bot, \sqsubseteq)$ into a domain (with least element $\bot$), called the flat domain determined by $X$. 
The product of two cpo’s \((D_1, \sqsubseteq_1)\) and \((D_2, \sqsubseteq_2)\) has underlying set

\[
D_1 \times D_2 = \{(d_1, d_2) \mid d_1 \in D_1 \& d_2 \in D_2\}
\]

and partial order \(\sqsubseteq\) defined by

\[
(d_1, d_2) \sqsubseteq (d'_1, d'_2) \overset{\text{def}}{{\iff}} d_1 \sqsubseteq_1 d'_1 \& d_2 \sqsubseteq_2 d'_2 .
\]
Lubs of chains are calculated componentwise:

\[
\bigsqcup_{n \geq 0} (d_{1,n}, d_{2,n}) = (\bigsqcup_{i \geq 0} d_{1,i}, \bigsqcup_{j \geq 0} d_{2,j}).
\]

If \((D_1, \sqsubseteq_1)\) and \((D_2, \sqsubseteq_2)\) are domains so is \((D_1 \times D_2, \sqsubseteq)\) and \(\bot_{D_1 \times D_2} = (\bot_{D_1}, \bot_{D_2})\).
Continuous functions of two arguments

**Proposition.** Let $D$, $E$, $F$ be cpo's. A function $f : (D \times E) \rightarrow F$ is monotone if and only if it is monotone in each argument separately:

\[
\forall d, d' \in D, e \in E. \ d \sqsubseteq d' \Rightarrow f(d, e) \sqsubseteq f(d', e)
\]
\[
\forall d \in D, e, e' \in E. \ e \sqsubseteq e' \Rightarrow f(d, e) \sqsubseteq f(d, e').
\]

Moreover, it is continuous if and only if it preserves lubs of chains in each argument separately:

\[
f(\bigsqcup_{m \geq 0} d_m, e) = \bigsqcup_{m \geq 0} f(d_m, e)
\]
\[
f(d, \bigsqcup_{n \geq 0} e_n) = \bigsqcup_{n \geq 0} f(d, e_n).
\]
• A couple of derived rules:

\[
\frac{x \sqsubseteq x' \quad y \sqsubseteq y'}{\quad f(x, y) \sqsubseteq f(x', y')} \quad (f \text{ monotone})
\]

\[
f(\bigsqcup_m x_m, \bigsqcup_n y_n) = \bigsqcup_k f(x_k, y_k)
\]
Function cpo’s and domains

Given cpo’s \((D, \sqsubseteq_D)\) and \((E, \sqsubseteq_E)\), the function cpo 
\((D \to E, \sqsubseteq)\) has underlying set

\[
(D \to E) \overset{\text{def}}{=} \{ f \mid f : D \to E \text{ is a continuous function} \}
\]

and partial order: \(f \sqsubseteq f' \overset{\text{def}}{\iff} \forall d \in D . f(d) \sqsubseteq_E f'(d).\)
Function cpo’s and domains

Given cpo’s \((\mathcal{D}, \sqsubseteq_D)\) and \((\mathcal{E}, \sqsubseteq_E)\), the function cpo \((\mathcal{D} \rightarrow \mathcal{E}, \sqsubseteq)\) has underlying set

\[
(\mathcal{D} \rightarrow \mathcal{E}) \overset{\text{def}}{=} \{ f \mid f : \mathcal{D} \rightarrow \mathcal{E} \text{ is a } \text{continuous function} \}
\]

and partial order: \(f \sqsubseteq f' \iff \forall d \in \mathcal{D}. f(d) \sqsubseteq_E f'(d)\).

- A derived rule:

\[
\begin{array}{c}
\text{\(f \sqsubseteq_{(\mathcal{D} \rightarrow \mathcal{E})} g\)} \\
\text{\(x \sqsubseteq_D y\)} \\
\hline
\text{\(f(x) \sqsubseteq g(y)\)}
\end{array}
\]
Lubs of chains are calculated ‘argumentwise’ (using lubs in $E$):

$$
\bigsqcup_{n \geq 0} f_n = \lambda d \in D. \ \bigsqcup_{n \geq 0} f_n(d).
$$

If $E$ is a domain, then so is $D \rightarrow E$ and $\bot_{D \rightarrow E}(d) = \bot_E$, all $d \in D$. 
Lubs of chains are calculated ‘argumentwise’ (using lubs in $E$):

$$
\bigsqcup_{n \geq 0} f_n = \lambda d \in D. \bigsqcup_{n \geq 0} f_n(d).
$$

- A derived rule:

$$( \bigsqcup_n f_n)(\bigsqcup_m x_m) = \bigsqcup_k f_k(x_k)$$

If $E$ is a domain, then so is $D \rightarrow E$ and $\perp_{D \rightarrow E}(d) = \perp_E$, all $d \in D$. 
Continuity of composition

For cpo’s $D, E, F$, the composition function

$$\circ : \left( (E \to F) \times (D \to E) \right) \to (D \to F)$$

declared by setting, for all $f \in (D \to E)$ and $g \in (E \to F)$,

$$g \circ f = \lambda d \in D. g(f(d))$$

is continuous.
Continuity of the fixpoint operator

Let $D$ be a domain.

By Tarski’s Fixed Point Theorem we know that each continuous function $f \in (D \to D)$ possesses a least fixed point, $\text{fix}(f) \in D$.

**Proposition.** The function

$$\text{fix} : (D \to D) \to D$$

is continuous.
Topic 4

Scott Induction
Scott’s Fixed Point Induction Principle

Let $f : D \to D$ be a continuous function on a domain $D$.

For any admissible subset $S \subseteq D$, to prove that the least fixed point of $f$ is in $S$, i.e. that

$$\text{fix}(f) \in S,$$

it suffices to prove

$$\forall d \in D \ (d \in S \Rightarrow f(d) \in S).$$
Chain-closed and admissible subsets

Let $D$ be a cpo. A subset $S \subseteq D$ is called chain-closed iff for all chains $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots$ in $D$

$$(\forall n \geq 0 . d_n \in S) \Rightarrow \left( \bigsqcup_{n \geq 0} d_n \right) \in S$$

If $D$ is a domain, $S \subseteq D$ is called admissible iff it is a chain-closed subset of $D$ and $\bot \in S$. 

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Chain-closed and admissible subsets

Let $D$ be a cpo. A subset $S \subseteq D$ is called chain-closed iff for all chains $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots$ in $D$

$$(\forall n \geq 0 . \ d_n \in S) \Rightarrow \left( \bigsqcup_{n \geq 0} d_n \right) \in S$$

If $D$ is a domain, $S \subseteq D$ is called admissible iff it is a chain-closed subset of $D$ and $\bot \in S$.

A property $\Phi(d)$ of elements $d \in D$ is called chain-closed (resp. admissible) iff $\{d \in D \mid \Phi(d)\}$ is a chain-closed (resp. admissible) subset of $D$. 
Let $D, E$ be cpos.

**Basic relations:**

- For every $d \in D$, the subset

$$\downarrow(d) \equiv \{ x \in D \mid x \sqsubseteq d \}$$

of $D$ is chain-closed.
Building chain-closed subsets (I)

Let $D, E$ be cpos.

**Basic relations:**

- For every $d \in D$, the subset
  \[ \downarrow(d) \overset{\text{def}}{=} \{ x \in D \mid x \sqsubseteq d \} \]
  of $D$ is chain-closed.

- The subsets
  \[ \{(x, y) \in D \times D \mid x \sqsubseteq y\} \]
  and
  \[ \{(x, y) \in D \times D \mid x = y\} \]
  of $D \times D$ are chain-closed.
Example (I): Least pre-fixed point property

Let $D$ be a domain and let $f : D \rightarrow D$ be a continuous function.

$$\forall d \in D. f(d) \sqsubseteq d \implies \text{fix}(f) \sqsubseteq d$$
Example (I): Least pre-fixed point property

Let $D$ be a domain and let $f : D \rightarrow D$ be a continuous function.

\[ \forall d \in D. \ f(d) \sqsubseteq d \implies \text{fix}(f) \sqsubseteq d \]

**Proof by Scott induction.**

Let $d \in D$ be a pre-fixed point of $f$. Then,

\[ x \in \downarrow(d) \implies x \sqsubseteq d \]
\[ \implies f(x) \sqsubseteq f(d) \]
\[ \implies f(x) \sqsubseteq d \]
\[ \implies f(x) \in \downarrow(d) \]

Hence,

\[ \text{fix}(f) \in \downarrow(d) \]
Inverse image:
Let $f : D \rightarrow E$ be a continuous function.
If $S$ is a chain-closed subset of $E$ then the inverse image

$$f^{-1}S = \{x \in D \mid f(x) \in S\}$$

is an chain-closed subset of $D$. 

Example (II)

Let $D$ be a domain and let $f, g : D \to D$ be continuous functions such that $f \circ g \sqsubseteq g \circ f$. Then,

\[ f(\bot) \sqsubseteq g(\bot) \implies \text{fix}(f) \sqsubseteq \text{fix}(g) . \]
Example (II)

Let $D$ be a domain and let $f, g : D \to D$ be continuous functions such that $f \circ g \sqsubseteq g \circ f$. Then,

$$f(\bot) \sqsubseteq g(\bot) \implies \text{fix}(f) \sqsubseteq \text{fix}(g).$$

*Proof by Scott induction.*

Consider the admissible property $\Phi(x) \equiv (f(x) \sqsubseteq g(x))$ of $D$.

Since

$$f(x) \sqsubseteq g(x) \Rightarrow g(f(x)) \sqsubseteq g(g(x)) \Rightarrow f(g(x)) \sqsubseteq g(g(x))$$

we have that

$$f(\text{fix}(g)) \sqsubseteq g(\text{fix}(g)).$$
Building chain-closed subsets (III)

Logical operations:

• If $S, T \subseteq D$ are chain-closed subsets of $D$ then
  \[ S \cup T \quad \text{and} \quad S \cap T \]
  are chain-closed subsets of $D$.

• If $\{ S_i \}_{i \in I}$ is a family of chain-closed subsets of $D$ indexed by a set $I$, then $\bigcap_{i \in I} S_i$ is a chain-closed subset of $D$.

• If a property $P(x, y)$ determines a chain-closed subset of $D \times E$, then the property $\forall x \in D. P(x, y)$ determines a chain-closed subset of $E$. 
Example (III): Partial correctness

Let $\mathcal{F} : \text{State} \rightarrow \text{State}$ be the denotation of

$$\text{while } X > 0 \text{ do } (Y := X \ast Y; X := X - 1) .$$

For all $x, y \geq 0$,

$$\mathcal{F}[X \mapsto x, Y \mapsto y] \downarrow$$

$$\implies \mathcal{F}[X \mapsto x, Y \mapsto y] = [X \mapsto 0, Y \mapsto x! \cdot y].$$
Recall that

\[ \mathcal{F} = \text{fix}(f) \]

where \( f : (\text{State} \rightarrow \text{State}) \rightarrow (\text{State} \rightarrow \text{State}) \) is given by

\[
f(w) = \lambda(x, y) \in \text{State}. \begin{cases} (x, y) & \text{if } x \leq 0 \\ w(x - 1, x \cdot y) & \text{if } x > 0 \end{cases}
\]
Proof by Scott induction.

We consider the admissible subset of \((State \rightarrow State)\) given by

\[
S = \left\{ w \mid \begin{array}{l}
\forall x, y \geq 0. \\
w[X \mapsto x, Y \mapsto y] \downarrow \\
\Rightarrow w[X \mapsto x, Y \mapsto y] = [X \mapsto 0, Y \mapsto x! \cdot y]
\end{array} \right\}
\]

and show that

\[
w \in S \implies f(w) \in S .
\]
Topic 5

PCF
PCF syntax

Types

$$\tau ::= \text{nat} \mid \text{bool} \mid \tau \rightarrow \tau$$
PCF syntax

Types

\[ \tau ::= \text{nat} \mid \text{bool} \mid \tau \rightarrow \tau \]

Expressions

\[ M ::= 0 \mid \text{succ}(M) \mid \text{pred}(M) \]
PCF syntax

Types

$$\tau ::= \text{nat} \mid \text{bool} \mid \tau \rightarrow \tau$$

Expressions

$$M ::= 0 \mid \text{succ}(M) \mid \text{pred}(M) \mid \text{true} \mid \text{false} \mid \text{zero}(M)$$
PCF syntax

Types

\[ \tau ::= \text{nat} \mid \text{bool} \mid \tau \to \tau \]

Expressions

\[ M ::= 0 \mid \text{succ}(M) \mid \text{pred}(M) \mid \text{true} \mid \text{false} \mid \text{zero}(M) \mid x \mid \text{if } M \text{ then } M \text{ else } M \]
PCF syntax

Types

\[ \tau ::= \text{nat} \mid \text{bool} \mid \tau \rightarrow \tau \]

Expressions

\[ M ::= 0 \mid \text{succ}(M) \mid \text{pred}(M) \]
\[ \mid \text{true} \mid \text{false} \mid \text{zero}(M) \]
\[ \mid x \mid \text{if } M \text{ then } M \text{ else } M \]
\[ \mid \text{fn } x : \tau . M \mid M \, M \mid \text{fix}(M) \]

where \( x \in \mathbb{V} \), an infinite set of variables.
PCF syntax

Types

\[ \tau ::= \text{nat} \mid \text{bool} \mid \tau \rightarrow \tau \]

Expressions

\[ M ::= 0 \mid \text{succ}(M) \mid \text{pred}(M) \]

\[ \mid \text{true} \mid \text{false} \mid \text{zero}(M) \]

\[ \mid x \mid \text{if } M \text{ then } M \text{ else } M \]

\[ \mid \text{fn } x : \tau . M \mid M \ M \mid \text{fix}(M) \]

where \( x \in \mathbb{V} \), an infinite set of variables.

**Technicality:** We identify expressions up to \( \alpha \)-conversion of bound variables (created by the \textit{fn} expression-former): by definition a PCF term is an \( \alpha \)-equivalence class of expressions.
PCF typing relation, \( \Gamma \vdash M : \tau \)

- \( \Gamma \) is a type environment, i.e. a finite partial function mapping variables to types (whose domain of definition is denoted \( \text{dom}(\Gamma) \))
- \( M \) is a term
- \( \tau \) is a type.
PCF typing relation, $\Gamma \vdash M : \tau$

- $\Gamma$ is a type environment, i.e. a finite partial function mapping variables to types (whose domain of definition is denoted $\text{dom}(\Gamma)$)
- $M$ is a term
- $\tau$ is a type.

Notation:

- $M : \tau$ means $M$ is closed and $\emptyset \vdash M : \tau$ holds.
- $\text{PCF}_\tau \overset{\text{def}}{=} \{ M \mid M : \tau \}.$
PCF typing relation (sample rules)

\[
\frac{\Gamma[x \mapsto \tau] \vdash M : \tau'}{\Gamma \vdash \text{fn} \ x : \tau \ . \ M : \tau \to \tau'} \quad \text{if } x \notin \text{dom}(\Gamma)
\]
PCF typing relation (sample rules)

\[\begin{align*}
\text{(fn)} & \quad \frac{\Gamma[x \mapsto \tau] \vdash M : \tau'}{\Gamma \vdash \text{fn } x : \tau . M : \tau \rightarrow \tau'} \quad \text{if } x \notin \text{dom}(\Gamma) \\
\text{(app)} & \quad \frac{\Gamma \vdash M_1 : \tau \rightarrow \tau' \quad \Gamma \vdash M_2 : \tau}{\Gamma \vdash M_1 \ M_2 : \tau'}
\end{align*}\]
PCF typing relation (sample rules)


delimiters:

\[
\begin{align*}
&\frac{\Gamma[x \mapsto \tau] \vdash M : \tau'}{
\Gamma \vdash \text{fn } x : \tau . M : \tau \rightarrow \tau'} \quad \text{if } x \notin \text{dom}(\Gamma)
\end{align*}
\]

\[
\begin{align*}
&\frac{\Gamma \vdash M_1 : \tau \rightarrow \tau' \quad \Gamma \vdash M_2 : \tau}{
\Gamma \vdash M_1 \, M_2 : \tau'}
\end{align*}
\]

\[
\begin{align*}
&\frac{\Gamma \vdash M : \tau \rightarrow \tau}{
\Gamma \vdash \text{fix}(M) : \tau}
\end{align*}
\]
Partial recursive functions in PCF

- Primitive recursion.

\[
\begin{align*}
h(x, 0) &= f(x) \\
h(x, y + 1) &= g(x, y, h(x, y))
\end{align*}
\]
Partial recursive functions in PCF

- Primitive recursion.

\[
\begin{align*}
h(x, 0) &= f(x) \\
h(x, y + 1) &= g(x, y, h(x, y))
\end{align*}
\]

- Minimisation.

\[
m(x) = \text{the least } y \geq 0 \text{ such that } k(x, y) = 0
\]
PCF evaluation relation

takes the form

\[ M \downarrow_\tau V \]

where

- \( \tau \) is a PCF type
- \( M, V \in \text{PCF}_\tau \) are closed PCF terms of type \( \tau \)
- \( V \) is a value,

\[
V ::= 0 \mid \text{succ}(V) \mid \text{true} \mid \text{false} \mid \text{fn } x : \tau . M.
\]
PCF evaluation (sample rules)

\[(\downarrow_{val}) \ V \downarrow_{\tau} V \quad (V \text{ a value of type } \tau)\]
PCF evaluation (sample rules)

\[(\downarrow_{\text{val}}) \quad V \downarrow_{\tau} V \quad (V \text{ a value of type } \tau)\]

\[(\downarrow_{\text{cbn}}) \quad \frac{M_1 \downarrow_{\tau \rightarrow \tau'} \textbf{fn} x : \tau \cdot M'_1 \quad M'_1[M_2/x] \downarrow_{\tau'} V}{M_1 \quad M_2 \downarrow_{\tau'} V}\]
PCF evaluation (sample rules)

\[ (\downarrow_{\text{val}}) \quad V \downarrow_{\tau} V \quad (V \text{ a value of type } \tau) \]

\[ (\downarrow_{\text{cbn}}) \quad M_1 \downarrow_{\tau \rightarrow \tau'} \quad \text{fn } x : \tau \cdot M'_1 \quad M'_1[M_2/x] \downarrow_{\tau'} V \]

\[ M_1 M_2 \downarrow_{\tau'} V \]

\[ (\downarrow_{\text{fix}}) \quad M \text{ fix}(M) \downarrow_{\tau} V \]

\[ \text{fix}(M) \downarrow_{\tau} V \]
Two phrases of a programming language are contextually equivalent if any occurrences of the first phrase in a complete program can be replaced by the second phrase without affecting the observable results of executing the program.
Given PCF terms $M_1$, $M_2$, PCF type $\tau$, and a type environment $\Gamma$, the relation $\Gamma \vdash M_1 \simeq_{\text{ctx}} M_2 : \tau$ is defined to hold iff

- Both the typings $\Gamma \vdash M_1 : \tau$ and $\Gamma \vdash M_2 : \tau$ hold.
- For all PCF contexts $C$ for which $C[M_1]$ and $C[M_2]$ are closed terms of type $\gamma$, where $\gamma = \text{nat}$ or $\gamma = \text{bool}$, and for all values $V : \gamma$,

$$C[M_1] \Downarrow_{\gamma} V \iff C[M_2] \Downarrow_{\gamma} V.$$
PCF denotational semantics — aims
PCF denotational semantics — aims

- PCF types $\tau \mapsto$ domains $\llbracket \tau \rrbracket$. 
PCF denotational semantics — aims

- PCF types $\tau \mapsto$ domains $[\tau]$.

- Closed PCF terms $M : \tau \mapsto$ elements $[M] \in [\tau]$.

  Denotations of open terms will be continuous functions.
PCF denotational semantics — aims

- PCF types $\tau \mapsto$ domains $[[\tau]]$.

- Closed PCF terms $M : \tau \mapsto$ elements $[[M]] \in [[\tau]]$.
  Denotations of open terms will be continuous functions.

- Compositionality.
  In particular: $[[M]] = [[M']] \Rightarrow [[C[M]]] = [[C[M']]$.
PCF denotational semantics — aims

- PCF types $\tau \mapsto$ domains $[[\tau]]$.

- Closed PCF terms $M : \tau \mapsto$ elements $[[M]] \in [[\tau]]$.
  Denotations of open terms will be continuous functions.

- Compositionality.
  In particular: $[[M]] = [[M']] \Rightarrow [[C[M]]] = [[C[M']]]$.

- Soundness.
  For any type $\tau$, $M \Downarrow_{\tau} V \Rightarrow [[M]] = [[V]]$. 
PCF denotational semantics — aims

- PCF types $\tau \mapsto$ domains $[\tau]$.

- Closed PCF terms $M : \tau \mapsto$ elements $[M] \in [\tau]$.
  Denotations of open terms will be continuous functions.

- Compositionality.
  \[ [M] = [M'] \Rightarrow [C[M]] = [C[M']] \]

- Soundness.
  For any type $\tau$, $M \downarrow_{\tau} V \Rightarrow [M] = [V]$.

- Adequacy.
  For $\tau = \text{bool}$ or $\text{nat}$, $[M] = [V] \in [\tau] \implies M \downarrow_{\tau} V$. 
Theorem. For all types $\tau$ and closed terms $M_1, M_2 \in \text{PCF}_\tau$, if $[[M_1]]$ and $[[M_2]]$ are equal elements of the domain $[[\tau]]$, then $M_1 \cong_{\text{ctx}} M_2 : \tau$. 
Theorem. For all types $\tau$ and closed terms $M_1, M_2 \in \text{PCF}_\tau$, if $[M_1]$ and $[M_2]$ are equal elements of the domain $[\tau]$, then $M_1 \simeq_{\text{ctx}} M_2 : \tau$.

Proof.

$C[M_1] \downarrow_{\text{nat}} V \Rightarrow [C[M_1]] = [V]$ \hspace{1cm} (soundness)

$\Rightarrow [C[M_2]] = [V]$ \hspace{1cm} (compositionality on $[M_1] = [M_2]$)

$\Rightarrow C[M_2] \downarrow_{\text{nat}} V$ \hspace{1cm} (adequacy)

and symmetrically. $\square$
To prove

\[ M_1 \simeq_{\text{ctx}} M_2 : \tau \]

it suffices to establish

\[ [M_1] = [M_2] \text{ in } \tau \]
Proof principle

To prove

\[ M_1 \simeq_{\text{ctx}} M_2 : \tau \]

it suffices to establish

\[ \llbracket M_1 \rrbracket = \llbracket M_2 \rrbracket \text{ in } \llbracket \tau \rrbracket \]

The proof principle is sound, but is it complete? That is, is equality in the denotational model also a necessary condition for contextual equivalence?
Topic 6

Denotational Semantics of PCF
To every typing judgement

\[ \Gamma \vdash M : \tau \]

we associate a continuous function

\[ [\Gamma \vdash M] : \llbracket \Gamma \rrbracket \rightarrow \llbracket \tau \rrbracket \]

between domains.
Denotational semantics of PCF types

\[ [nat] \overset{\text{def}}{=} \mathbb{N}_\perp \] (flat domain)

\[ [bool] \overset{\text{def}}{=} \mathbb{B}_\perp \] (flat domain)

where \( \mathbb{N} = \{0, 1, 2, \ldots\} \) and \( \mathbb{B} = \{true, false\} \).
Denotational semantics of PCF types

\[ [nat] \overset{\text{def}}{=} N_\bot \] (flat domain)

\[ [bool] \overset{\text{def}}{=} B_\bot \] (flat domain)

\[ [\tau \rightarrow \tau'] \overset{\text{def}}{=} [\tau] \rightarrow [\tau'] \] (function domain).

where \( N = \{0, 1, 2, \ldots\} \) and \( B = \{true, false\} \).
Denotational semantics of PCF type environments

\[ [\Gamma] \overset{\text{def}}{=} \prod_{x \in \text{dom}(\Gamma)} [\Gamma(x)] \]  
\((\Gamma\text{-environments})\)
Denotational semantics of PCF type environments

\[ [\Gamma] \overset{\text{def}}{=} \prod_{x \in \text{dom}(\Gamma)} [\Gamma(x)] \quad (\Gamma\text{-environments}) \]

= the domain of partial functions \( \rho \) from variables to domains such that \( \text{dom}(\rho) = \text{dom}(\Gamma) \) and \( \rho(x) \in [\Gamma(x)] \) for all \( x \in \text{dom}(\Gamma) \)
Denotational semantics of PCF type environments

$$[[\Gamma]] \overset{\text{def}}{=} \prod_{x \in \text{dom}(\Gamma)} [[\Gamma(x)]] \quad (\Gamma\text{-environments})$$

= the domain of partial functions $\rho$ from variables to domains such that $\text{dom}(\rho) = \text{dom}(\Gamma)$ and $\rho(x) \in [[\Gamma(x)]]$ for all $x \in \text{dom}(\Gamma)$

Example:

1. For the empty type environment $\emptyset$,

$$[[\emptyset]] = \{ \bot \}$$

where $\bot$ denotes the unique partial function with $\text{dom}(\bot) = \emptyset$. 
2. \[\llbracket \langle x \mapsto \tau \rangle \rrbracket = \left( \{ x \} \to \llbracket \tau \rrbracket \right)\]
2. $\llbracket (x \mapsto \tau) \rrbracket = \left( \{ x \} \rightarrow [\tau] \right) \cong [\tau]$
2. \[ \llbracket \langle x \mapsto \tau \rangle \rrbracket = \left( \{ x \} \rightarrow \llbracket \tau \rrbracket \right) \cong \llbracket \tau \rrbracket \]

3. 
\[ \llbracket \langle x_1 \mapsto \tau_1, \ldots, x_n \mapsto \tau_n \rangle \rrbracket \]
\[ \cong \left( \{ x_1 \} \rightarrow \llbracket \tau_1 \rrbracket \right) \times \ldots \times \left( \{ x_n \} \rightarrow \llbracket \tau_n \rrbracket \right) \]
\[ \cong \llbracket \tau_1 \rrbracket \times \ldots \times \llbracket \tau_n \rrbracket \]
Denotational semantics of PCF terms, I

\[ [\Gamma \vdash 0](\rho) \overset{\text{def}}{=} 0 \in [nat] \]

\[ [\Gamma \vdash \text{true}](\rho) \overset{\text{def}}{=} \text{true} \in [bool] \]

\[ [\Gamma \vdash \text{false}](\rho) \overset{\text{def}}{=} \text{false} \in [bool] \]
Denotational semantics of PCF terms, I

\[
[\Gamma \vdash 0](\rho) \overset{\text{def}}{=} 0 \in [\text{nat}]
\]

\[
[\Gamma \vdash \text{true}](\rho) \overset{\text{def}}{=} \text{true} \in [\text{bool}]
\]

\[
[\Gamma \vdash \text{false}](\rho) \overset{\text{def}}{=} \text{false} \in [\text{bool}]
\]

\[
[\Gamma \vdash x](\rho) \overset{\text{def}}{=} \rho(x) \in [\Gamma(x)] \quad (x \in \text{dom}(\Gamma))
\]
\[
\llbracket \Gamma \vdash \textbf{succ}(M) \rrbracket(\rho) = \begin{cases} 
\llbracket \Gamma \vdash M \rrbracket(\rho) + 1 & \text{if } \llbracket \Gamma \vdash M \rrbracket(\rho) \neq \bot \\
\bot & \text{if } \llbracket \Gamma \vdash M \rrbracket(\rho) = \bot
\end{cases}
\]
\[
\left[ \Gamma \vdash \text{succ}(M) \right](\rho)
\]
\[
\text{def } \begin{cases} 
\left[ \Gamma \vdash M \right](\rho) + 1 & \text{if } \left[ \Gamma \vdash M \right](\rho) \neq \bot \\
\bot & \text{if } \left[ \Gamma \vdash M \right](\rho) = \bot
\end{cases}
\]

\[
\left[ \Gamma \vdash \text{pred}(M) \right](\rho)
\]
\[
\text{def } \begin{cases} 
\left[ \Gamma \vdash M \right](\rho) - 1 & \text{if } \left[ \Gamma \vdash M \right](\rho) > 0 \\
\bot & \text{if } \left[ \Gamma \vdash M \right](\rho) = 0, \bot
\end{cases}
\]
\[
[\Gamma \vdash \text{succ}(M)](\rho) \overset{\text{def}}{=} \begin{cases} 
[\Gamma \vdash M](\rho) + 1 & \text{if } [\Gamma \vdash M](\rho) \neq \bot \\
\bot & \text{if } [\Gamma \vdash M](\rho) = \bot
\end{cases}
\]

\[
[\Gamma \vdash \text{pred}(M)](\rho) \overset{\text{def}}{=} \begin{cases} 
[\Gamma \vdash M](\rho) - 1 & \text{if } [\Gamma \vdash M](\rho) > 0 \\
\bot & \text{if } [\Gamma \vdash M](\rho) = 0, \bot
\end{cases}
\]

\[
[\Gamma \vdash \text{zero}(M)](\rho) \overset{\text{def}}{=} \begin{cases} 
\text{true} & \text{if } [\Gamma \vdash M](\rho) = 0 \\
\text{false} & \text{if } [\Gamma \vdash M](\rho) > 0 \\
\bot & \text{if } [\Gamma \vdash M](\rho) = \bot
\end{cases}
\]
Denotational semantics of PCF terms, III

\[
\llbracket \Gamma \vdash \mathbf{if} \ M_1 \ \mathbf{then} \ M_2 \ \mathbf{else} \ M_3 \rrbracket (\rho) \defeq \begin{cases} 
\llbracket \Gamma \vdash M_2 \rrbracket (\rho) & \text{if } \llbracket \Gamma \vdash M_1 \rrbracket (\rho) = \text{true} \\
\llbracket \Gamma \vdash M_3 \rrbracket (\rho) & \text{if } \llbracket \Gamma \vdash M_1 \rrbracket (\rho) = \text{false} \\
\bot & \text{if } \llbracket \Gamma \vdash M_1 \rrbracket (\rho) = \bot
\end{cases}
\]
Denotational semantics of PCF terms, III

\[
\llbracket \Gamma \vdash \text{if } M_1 \text{ then } M_2 \text{ else } M_3 \rrbracket (\rho) \defeq \begin{cases} 
\llbracket \Gamma \vdash M_2 \rrbracket (\rho) & \text{if } \llbracket \Gamma \vdash M_1 \rrbracket (\rho) = \text{true} \\
\llbracket \Gamma \vdash M_3 \rrbracket (\rho) & \text{if } \llbracket \Gamma \vdash M_1 \rrbracket (\rho) = \text{false} \\
\bot & \text{if } \llbracket \Gamma \vdash M_1 \rrbracket (\rho) = \bot
\end{cases}
\]

\[
\llbracket \Gamma \vdash M_1 \ M_2 \rrbracket (\rho) \defeq \left( \llbracket \Gamma \vdash M_1 \rrbracket (\rho) \right) \left( \llbracket \Gamma \vdash M_2 \rrbracket (\rho) \right)
\]
Denotational semantics of PCF terms, IV

\[
\begin{align*}
[\Gamma \vdash \text{fn } x : \tau . \ M](\rho) & \overset{\text{def}}{=} \lambda d \in [\tau] . [\Gamma[x \mapsto \tau] \vdash M](\rho[x \mapsto d]) \\
(x \notin \text{dom}(\Gamma))
\end{align*}
\]

\textbf{NB:} \( \rho[x \mapsto d] \in [\Gamma[x \mapsto \tau]] \) is the function mapping \( x \) to \( d \in [\tau] \) and otherwise acting like \( \rho \).
Denotational semantics of PCF terms, V

\[
\llbracket \Gamma \vdash \text{fix}(M) \rrbracket(\rho) \overset{\text{def}}{=} \text{fix}(\llbracket \Gamma \vdash M \rrbracket(\rho))
\]

Recall that \text{fix} is the function assigning least fixed points to continuous functions.
Denotational semantics of PCF

**Proposition.** For all typing judgements $\Gamma \vdash M : \tau$, the denotation

$$[\Gamma \vdash M] : [\Gamma] \rightarrow [\tau]$$

is a well-defined continuous function.
Denotations of closed terms

For a closed term $M \in \text{PCF}_\tau$, we get

$$[[\emptyset \vdash M]] : [[\emptyset]] \rightarrow [[\tau]]$$

and, since $[[\emptyset]] = \{\bot\}$, we have

$$[[M]] \overset{\text{def}}{=} [[\emptyset \vdash M]](\bot) \in [[\tau]] \quad (M \in \text{PCF}_\tau)$$
Compositionality

**Proposition.** For all typing judgements \( \Gamma \vdash M : \tau \) and \( \Gamma \vdash M' : \tau \), and all contexts \( C[\_] \) such that \( \Gamma' \vdash C[M] : \tau' \) and \( \Gamma' \vdash C[M'] : \tau' \),

\[
\text{if } [\Gamma \vdash M] = [\Gamma \vdash M'] : [\Gamma] \rightarrow [\tau] \\
\text{then } [\Gamma' \vdash C[M]] = [\Gamma' \vdash C[M]] : [\Gamma'] \rightarrow [\tau']
\]
Soundness

Proposition. For all closed terms $M, V \in \text{PCF}_\tau$,

$$\text{if } M \Downarrow_\tau V \text{ then } [M] = [V] \in [\tau].$$
Substitution property

**Proposition.** Suppose that $\Gamma \vdash M : \tau$ and that $\Gamma[x \mapsto \tau] \vdash M' : \tau'$, so that we also have $\Gamma \vdash M'[M/x] : \tau'$.

Then,

$$\left[\left[\Gamma \vdash M'[M/x]\right]\right](\rho)$$

$$= \left[\left[\Gamma[x \mapsto \tau] \vdash M'\right]\right](\rho[x \mapsto \left[\Gamma \vdash M\right](\rho)])$$

for all $\rho \in \left[\left[\Gamma\right]\right]$. 
Substitution property

**Proposition.** Suppose that $\Gamma \vdash M : \tau$ and that $\Gamma [x \mapsto \tau] \vdash M' : \tau'$, so that we also have $\Gamma \vdash M'[M/x] : \tau'$. Then,

$$\left[ \Gamma \vdash M'[M/x] \right] (\rho) = \left[ \Gamma [x \mapsto \tau] \vdash M' \right] (\rho [x \mapsto \left[ \Gamma \vdash M \right](\rho)])$$

for all $\rho \in \left[ \Gamma \right]$.

In particular when $\Gamma = \emptyset$, $\left[ \langle x \mapsto \tau \rangle \vdash M' \right] : \left[ \tau \right] \rightarrow \left[ \tau' \right]$ and

$$\left[ M'[M/x] \right] = \left[ \langle x \mapsto \tau \rangle \vdash M' \right] \left( \left[ M \right] \right)$$
Topic 7

Relating Denotational and Operational Semantics
Adequacy

For any closed PCF terms $M$ and $V$ of ground type $\gamma \in \{nat, bool\}$ with $V$ a value

$$[M] = [V] \in [\gamma] \implies M \Downarrow_{\gamma} V.$$
Adequacy

For any closed PCF terms $M$ and $V$ of \textit{ground} type $\gamma \in \{\text{nat}, \text{bool}\}$ with $V$ a value

$$[M] = [V] \in \lfloor \gamma \rfloor \implies M \downarrow_\gamma V.$$ 

\textbf{NB.} Adequacy does not hold at function types
Adequacy

For any closed PCF terms $M$ and $V$ of ground type $\gamma \in \{\text{nat, bool}\}$ with $V$ a value

\[ [M] = [V] \in [\gamma] \implies M \downarrow_{\gamma} V. \]

**NB.** Adequacy does not hold at function types:

\[
[\text{fn } x : \tau. (\text{fn } y : \tau. y) \ x] = [\text{fn } x : \tau. x] : [\tau] \rightarrow [\tau]
\]
Adequacy

For any closed PCF terms $M$ and $V$ of ground type $\gamma \in \{\text{nat}, \text{bool}\}$ with $V$ a value

$$\llbracket M \rrbracket = \llbracket V \rrbracket \in \llbracket \gamma \rrbracket \implies M \downarrow_{\gamma} V .$$

**NB.** Adequacy does not hold at function types:

$$\llbracket \text{fn } x : \tau. (\text{fn } y : \tau. y) \ x \rrbracket = \llbracket \text{fn } x : \tau. x \rrbracket : \llbracket \tau \rrbracket \rightarrow \llbracket \tau \rrbracket$$

but

$$\text{fn } x : \tau. (\text{fn } y : \tau. y) \ x \nmid_{\tau \rightarrow \tau} \text{fn } x : \tau. x$$
Adequacy proof idea
Adequacy proof idea

1. We cannot proceed to prove the adequacy statement by a straightforward induction on the structure of terms.

   Consider $M$ to be $M_1 M_2$, $\text{fix}(M')$. 
Adequacy proof idea

1. We cannot proceed to prove the adequacy statement by a straightforward induction on the structure of terms.

   ▶ Consider $M$ to be $M_1 M_2$, $\text{fix}(M')$.

2. So we proceed to prove a stronger statement that applies to terms of arbitrary types and implies adequacy.
Adequacy proof idea

1. We cannot proceed to prove the adequacy statement by a straightforward induction on the structure of terms.

   - Consider $M$ to be $M_1 M_2$, $\text{fix}(M')$.

2. So we proceed to prove a stronger statement that applies to terms of arbitrary types and implies adequacy.

   This statement roughly takes the form:

   $$\boxed{\llbracket M \rrbracket \trianglelefteq_\tau M \text{ for all types } \tau \text{ and all } M \in \text{PCF}_\tau}$$

   where the *formal approximation relations*

   $$\trianglelefteq_\tau \subseteq \llbracket \tau \rrbracket \times \text{PCF}_\tau$$

   are *logically* chosen to allow a proof by induction.
We want that, for $\gamma \in \{\text{nat}, \text{bool}\}$,

\[
[M] \triangleleft_\gamma M \implies \forall V ([M] = [V] \implies M \downarrow_\gamma V)
\]

adequacy
Definition of \( d \triangleleft_{\gamma} M \) (\( d \in \left[ \gamma \right], M \in \text{PCF}_\gamma \))
for \( \gamma \in \{ \text{nat, bool} \} \)

\[
n \triangleleft_{\text{nat}} M \quad \overset{\text{def}}{\iff} \quad (n \in \mathbb{N} \Rightarrow M \downarrow_{\text{nat}} \text{succ}^n(0))
\]

\[
b \triangleleft_{\text{bool}} M \quad \overset{\text{def}}{\iff} \quad (b = \text{true} \Rightarrow M \downarrow_{\text{bool}} \text{true})
\& (b = \text{false} \Rightarrow M \downarrow_{\text{bool}} \text{false})
\]
Proof of: $[M] \triangleleft_{\gamma} M$ implies adequacy

Case $\gamma = nat$.

$[M] = [V]$

$\Rightarrow [M] = [\text{succ}^n(0)]$ for some $n \in \mathbb{N}$

$\Rightarrow n = [M] \triangleleft_{\gamma} M$

$\Rightarrow M \Downarrow \text{succ}^n(0)$ by definition of $\triangleleft_{nat}$

Case $\gamma = bool$ is similar.
We want to be able to proceed by induction.

Consider the case $M = M_1 M_2$. 

\[ \sim logical \text{ definition} \]
Definition of

\[ f \triangleleft_{\tau \rightarrow \tau'} M \ (f \in ([\tau] \rightarrow [\tau']), \ M \in \text{PCF}_{\tau \rightarrow \tau'}) \]
Definition of

\[ f \triangleleft_{\tau \rightarrow \tau'} M \quad (f \in ([\tau] \rightarrow [\tau']), \ M \in \text{PCF}_{\tau \rightarrow \tau'}) \]

\[ f \triangleleft_{\tau \rightarrow \tau'} M \]

\[ \overset{\text{def}}{\iff} \quad \forall \ x \in [\tau], \ N \in \text{PCF}_\tau \]

\[ (x \triangleleft_{\tau} N \ \Rightarrow \ f(x) \triangleleft_{\tau'} M \ N) \]
We want to be able to proceed by induction.

Consider the case \( M = \text{fix}(M') \).

\( \sim \) admissibility property
Admissibility property

Lemma. For all types $\tau$ and $M \in \text{PCF}_\tau$, the set

$$\{ d \in [\tau] \mid d \prec_\tau M \}$$

is an admissible subset of $[\tau]$. 
Further properties

Lemma. For all types $\tau$, elements $d, d' \in \llbracket \tau \rrbracket$, and terms $M, N, V \in \text{PCF}_\tau$,

1. If $d \sqsubseteq d'$ and $d' \triangleleft^\tau M$ then $d \triangleleft^\tau M$.

2. If $d \triangleleft^\tau M$ and $\forall V \ (M \Downarrow^\tau V \implies N \Downarrow^\tau V)$ then $d \triangleleft^\tau N$. 
We want to be able to proceed by induction.

Consider the case $M = \text{fn } x : \tau . M'$.

$\rightsquigarrow$ *substitutivity* property for open terms
Theorem. For all \( \Gamma = \langle x_1 \mapsto \tau_1, \ldots, x_n \mapsto \tau_n \rangle \) and all \( \Gamma \vdash M : \tau \), if \( d_1 \triangleleft_{\tau_1} M_1, \ldots, d_n \triangleleft_{\tau_n} M_n \) then 
\[
\llbracket \Gamma \vdash M \rrbracket \llbracket x_1 \mapsto d_1, \ldots, x_n \mapsto d_n \rrbracket \triangleleft_{\tau} M[M_1/x_1, \ldots, M_n/x_n].
\]
Fundamental property

Theorem. For all \( \Gamma = \langle x_1 \mapsto \tau_1, \ldots, x_n \mapsto \tau_n \rangle \) and all \( \Gamma \vdash M : \tau \), if \( d_1 \triangleleft_{\tau_1} M_1, \ldots, d_n \triangleleft_{\tau_n} M_n \) then

\[
\llbracket \Gamma \vdash M \rrbracket [x_1 \mapsto d_1, \ldots, x_n \mapsto d_n] \triangleleft \tau M[M_1/x_1, \ldots, M_n/x_n].
\]

NB. The case \( \Gamma = \emptyset \) reduces to

\[
\llbracket M \rrbracket \triangleleft \tau M
\]

for all \( M \in \text{PCF}_\tau \).
Fundamental property of the relations $\bowtie_{\tau}$

Proposition. If $\Gamma \vdash M : \tau$ is a valid PCF typing, then for all $\Gamma$-environments $\rho$ and all $\Gamma$-substitutions $\sigma$

$$\rho \bowtie_{\Gamma} \sigma \implies [\Gamma \vdash M](\rho) \bowtie_{\tau} M[\sigma]$$

- $\rho \bowtie_{\Gamma} \sigma$ means that $\rho(x) \bowtie_{\Gamma(x)} \sigma(x)$ holds for each $x \in \text{dom}(\Gamma)$.

- $M[\sigma]$ is the PCF term resulting from the simultaneous substitution of $\sigma(x)$ for $x$ in $M$, each $x \in \text{dom}(\Gamma)$. 
Contextual preorder between PCF terms

Given PCF terms $M_1, M_2$, PCF type $\tau$, and a type environment $\Gamma$, the relation $\Gamma \vdash M_1 \leq_{ctx} M_2 : \tau$ is defined to hold iff

- Both the typings $\Gamma \vdash M_1 : \tau$ and $\Gamma \vdash M_2 : \tau$ hold.

- For all PCF contexts $C$ for which $C[M_1]$ and $C[M_2]$ are closed terms of type $\gamma$, where $\gamma = \text{nat}$ or $\gamma = \text{bool}$, and for all values $V \in \text{PCF}_\gamma$,

  $$C[M_1] \Downarrow_\gamma V \implies C[M_2] \Downarrow_\gamma V.$$
Extensionality properties of $\leq_{\text{ctx}}$

At a ground type $\gamma \in \{\text{bool, nat}\}$,

$$M_1 \leq_{\text{ctx}} M_2 : \gamma \text{ holds if and only if } \forall V \in \text{PCF}_\gamma (M_1 \downarrow_\gamma V \implies M_2 \downarrow_\gamma V).$$

At a function type $\tau \rightarrow \tau'$,

$$M_1 \leq_{\text{ctx}} M_2 : \tau \rightarrow \tau' \text{ holds if and only if } \forall M \in \text{PCF}_\tau (M_1 M \leq_{\text{ctx}} M_2 M : \tau').$$
Topic 8

Full Abstraction
**Proof principle**

For all types $\tau$ and closed terms $M_1, M_2 \in \text{PCF}_\tau$,

$$[[M_1]] = [[M_2]] \text{ in } [\tau] \implies M_1 \simeq_{\text{ctx}} M_2 : \tau .$$

Hence, to prove

$$M_1 \simeq_{\text{ctx}} M_2 : \tau$$

it suffices to establish

$$[[M_1]] = [[M_2]] \text{ in } [\tau] .$$
A denotational model is said to be *fully abstract* whenever denotational equality characterises contextual equivalence.
A denotational model is said to be \textit{fully abstract} whenever denotational equality characterises contextual equivalence.

The domain model of \textit{PCF} is \textit{not} fully abstract.

In other words, there are contextually equivalent \textit{PCF} terms with different denotations.
Failure of full abstraction, idea

We will construct two closed terms

\[ T_1, T_2 \in \text{PCF}(\text{bool} \rightarrow (\text{bool} \rightarrow \text{bool})) \rightarrow \text{bool} \]

such that

\[ T_1 \cong_{\text{ctx}} T_2 \]

and

\[ [T_1] \neq [T_2] \]
We achieve $T_1 \simeq_{ctx} T_2$ by making sure that

$$\forall M \in \text{PCF}_{\text{bool} \to (\text{bool} \to \text{bool})} \left( T_1 M \nvdash_{\text{bool}} \& T_2 M \nvdash_{\text{bool}} \right)$$
We achieve \( T_1 \simeq_{\text{ctx}} T_2 \) by making sure that

\[
\forall M \in \text{PCF}_{\text{bool} \to (\text{bool} \to \text{bool})} \left( T_1 \ M \not\downarrow_{\text{bool}} \land T_2 \ M \not\downarrow_{\text{bool}} \right)
\]

Hence,

\[
[T_1](\llbracket M \rrbracket) = \bot = [T_2](\llbracket M \rrbracket)
\]

for all \( M \in \text{PCF}_{\text{bool} \to (\text{bool} \to \text{bool})} \).
We achieve $T_1 \cong_{\text{ctx}} T_2$ by making sure that

$$\forall M \in \text{PCF}_{\text{bool} \to (\text{bool} \to \text{bool})} \left( T_1 M \not\downarrow_{\text{bool}} \& T_2 M \not\downarrow_{\text{bool}} \right)$$

Hence,

$$[[T_1]]([[M]]) = \bot = [[T_2]]([[M]])$$

for all $M \in \text{PCF}_{\text{bool} \to (\text{bool} \to \text{bool})}$.

We achieve $[[T_1]] \neq [[T_2]]$ by making sure that

$$[[T_1]](\text{por}) \neq [[T_2]](\text{por})$$

for some non-definable continuous function

$$\text{por} \in (\mathbb{B}_\bot \to (\mathbb{B}_\bot \to \mathbb{B}_\bot))$$.
Parallel-or function

is the unique continuous function \( \text{por} : \mathbb{B}_\bot \rightarrow (\mathbb{B}_\bot \rightarrow \mathbb{B}_\bot) \) such that

\[
\begin{align*}
\text{por true } \bot & = \text{ true} \\
\text{por } \bot \text{ true} & = \text{ true} \\
\text{por false false} & = \text{ false}
\end{align*}
\]
Parallel-or function

is the unique continuous function \( \text{por} : \mathbb{B}_\bot \rightarrow (\mathbb{B}_\bot \rightarrow \mathbb{B}_\bot) \) such that

\[
\begin{align*}
\text{por} \; \text{true} \; \bot &= \text{true} \\
\text{por} \; \bot \; \text{true} &= \text{true} \\
\text{por} \; \text{false} \; \text{false} &= \text{false}
\end{align*}
\]

In which case, it necessarily follows by monotonicity that

\[
\begin{align*}
\text{por} \; \text{true} \; \text{true} &= \text{true} & \text{por} \; \text{false} \; \bot &= \bot \\
\text{por} \; \text{true} \; \text{false} &= \text{true} & \text{por} \; \bot \; \text{false} &= \bot \\
\text{por} \; \text{false} \; \text{true} &= \text{true} & \text{por} \; \bot \; \bot &= \bot
\end{align*}
\]
Undefinability of parallel-or

Proposition. There is no closed PCF term

\[ P : \text{bool} \rightarrow (\text{bool} \rightarrow \text{bool}) \]

satisfying

\[ [P] = \text{por} : \mathbb{B}_\bot \rightarrow (\mathbb{B}_\bot \rightarrow \mathbb{B}_\bot) . \]
Parallel-or test functions
Parallel-or test functions

For $i = 1, 2$ define

$$T_i \overset{\text{def}}{=} \text{fn } f : \text{bool} \to (\text{bool} \to \text{bool}).$$

if $(f \text{ true } \Omega)$ then
  if $(f \Omega \text{ true})$ then
    if $(f \text{ false false})$ then $\Omega$ else $B_i$
  else $\Omega$
else $\Omega$

where $B_1 \overset{\text{def}}{=} \text{true}$, $B_2 \overset{\text{def}}{=} \text{false}$,
and $\Omega \overset{\text{def}}{=} \text{fix}(\text{fn } x : \text{bool} . x)$. 
Failure of full abstraction

Proposition.

\[ T_1 \simeq_{\text{ctx}} T_2 : (\text{bool} \to (\text{bool} \to \text{bool})) \to \text{bool} \]

\[ [T_1] \neq [T_2] \in (\mathbb{B}_\bot \to (\mathbb{B}_\bot \to \mathbb{B}_\bot)) \to \mathbb{B}_\bot \]
Expressions\[ M ::= \cdots \mid \textsf{por}(M, M) \]

Typing\[ \Gamma \vdash M_1 : \text{bool} \quad \Gamma \vdash M_2 : \text{bool} \]
\[ \Gamma \vdash \textsf{por}(M_1, M_2) : \text{bool} \]

Evaluation\[ M_1 \downarrow_{\text{bool}} \text{true} \quad M_2 \downarrow_{\text{bool}} \text{true} \]
\[ \text{por}(M_1, M_2) \downarrow_{\text{bool}} \text{true} \quad \text{por}(M_1, M_2) \downarrow_{\text{bool}} \text{true} \]
\[ M_1 \downarrow_{\text{bool}} \text{false} \quad M_2 \downarrow_{\text{bool}} \text{false} \]
\[ \text{por}(M_1, M_2) \downarrow_{\text{bool}} \text{false} \]
Plotkin’s full abstraction result

The denotational semantics of PCF+por is given by extending that of PCF with the clause

$$[[\Gamma \vdash \text{por}(M_1, M_2)](\rho) \overset{\text{def}}{=} \text{por}([[\Gamma \vdash M_1])(\rho)) (\Gamma \vdash M_2)](\rho))$$

This denotational semantics is fully abstract for contextual equivalence of PCF+por terms:

$$\Gamma \vdash M_1 \cong_{\text{ctx}} M_2 : \tau \iff [[\Gamma \vdash M_1] = [[\Gamma \vdash M_2].$$