Topic 6

Denotational Semantics of PCF

Denotational semantics of PCF

To every typing judgement

$$\Gamma \vdash M : \tau$$

we associate a continuous function

$$[\![\Gamma \vdash M]\!] : [\![\Gamma]\!] \to [\![\tau]\!]$$

between domains.

Denotational semantics of PCF types

$$[\![nat]\!] \stackrel{\text{def}}{=} \mathbb{N}_\perp$$ (flat domain)

$$[\![bool]\!] \stackrel{\text{def}}{=} \mathbb{B}_\perp$$ (flat domain)

where $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{B} = \{true, false\}$.

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$$[\![\tau \to \tau']\!] \stackrel{\text{def}}{=} [\![\tau]\!] \to [\![\tau']\!]$$ (function domain).

where $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{B} = \{true, false\}$.

Denotational semantics of PCF type environments

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Denotational semantics of PCF type environments

$$\llbracket \Gamma \rrbracket \stackrel{\text{def}}{=} \prod_{x \in dom(\Gamma)} \llbracket \Gamma(x) \rrbracket$$ (Γ-environments)

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Example:

1. For the empty type environment $\emptyset$,

$$\llbracket \emptyset \rrbracket = \{ \perp \}$$

where $\perp$ denotes the unique partial function with $dom(\perp) = \emptyset$.

2. $[\![\langle x \mapsto \tau \rangle]\!] \;=\; \big(\{\, x \,\} \to [\![\tau]\!]\big)$

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3.

$$
\begin{aligned}
& [\![\langle x_1 \mapsto \tau_1, \ldots, x_n \mapsto \tau_n \rangle]\!] \\
& \quad \cong (\{ x_1 \} \to [\![\tau_1]\!]) \times \ldots \times (\{ x_n \} \to [\![\tau_n]\!]) \\
& \quad \cong [\![\tau_1]\!] \times \ldots \times [\![\tau_n]\!]
\end{aligned}
$$

Denotational semantics of PCF terms, I

$$[\![\Gamma \vdash \mathbf{0}]\!](\rho) \stackrel{\text{def}}{=} 0 \in [\![nat]\!]$$

$$[\![\Gamma \vdash \mathbf{true}]\!](\rho) \stackrel{\text{def}}{=} true \in [\![bool]\!]$$

$$[\![\Gamma \vdash \mathbf{false}]\!](\rho) \stackrel{\text{def}}{=} false \in [\![bool]\!]$$

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$$[\![\Gamma \vdash x]\!](\rho) \stackrel{\text{def}}{=} \rho(x) \in [\![\Gamma(x)]\!] \qquad \big(x \in dom(\Gamma)\big)$$

Denotational semantics of PCF terms, II

$$[\![\Gamma \vdash \mathbf{succ}(M)]\!](\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} [\![\Gamma \vdash M]\!](\rho) + 1 & \text{if } [\![\Gamma \vdash M]\!](\rho) \neq \bot \\ \bot & \text{if } [\![\Gamma \vdash M]\!](\rho) = \bot \end{cases}$$

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$$[\![\Gamma \vdash \mathbf{pred}(M)]\!](\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} [\![\Gamma \vdash M]\!](\rho) - 1 & \text{if } [\![\Gamma \vdash M]\!](\rho) > 0 \\ \bot & \text{if } [\![\Gamma \vdash M]\!](\rho) = 0, \bot \end{cases}$$

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$$[\![\Gamma \vdash \mathbf{zero}(M)]\!](\rho) \stackrel{\text{def}}{=} \begin{cases} \mathit{true} & \text{if } [\![\Gamma \vdash M]\!](\rho) = 0 \\ \mathit{false} & \text{if } [\![\Gamma \vdash M]\!](\rho) > 0 \\ \bot & \text{if } [\![\Gamma \vdash M]\!](\rho) = \bot \end{cases}$$

Denotational semantics of PCF terms, III

$$[\![\Gamma \vdash \mathbf{if}\ M_1\ \mathbf{then}\ M_2\ \mathbf{else}\ M_3]\!](\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} [\![\Gamma \vdash M_2]\!](\rho) & \text{if } [\![\Gamma \vdash M_1]\!](\rho) = \mathit{true} \\ [\![\Gamma \vdash M_3]\!](\rho) & \text{if } [\![\Gamma \vdash M_1]\!](\rho) = \mathit{false} \\ \bot & \text{if } [\![\Gamma \vdash M_1]\!](\rho) = \bot \end{cases}$$

Denotational semantics of PCF terms, III

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$$[\![\Gamma \vdash M_1\ M_2]\!](\rho) \stackrel{\text{def}}{=} \big([\![\Gamma \vdash M_1]\!](\rho)\big)\ \big([\![\Gamma \vdash M_2]\!](\rho)\big)$$

Denotational semantics of PCF terms, IV

$$\llbracket \Gamma \vdash \mathbf{fn}\, x : \tau \,.\, M \rrbracket(\rho) \stackrel{\text{def}}{=} \lambda d \in \llbracket \tau \rrbracket \,.\, \llbracket \Gamma[x \mapsto \tau] \vdash M \rrbracket(\rho[x \mapsto d]) \qquad (x \notin dom(\Gamma))$$

NB: $\rho[x \mapsto d] \in \llbracket \Gamma[x \mapsto \tau] \rrbracket$ is the function mapping x to $d \in \llbracket \tau \rrbracket$ and otherwise acting like ρ.

Denotational semantics of PCF terms, V

$$[\![\Gamma \vdash \mathbf{fix}(M)]\!](\rho) \stackrel{\text{def}}{=} \mathit{fix}([\![\Gamma \vdash M]\!](\rho))$$

Recall that fix is the function assigning least fixed points to continuous functions.

Denotational semantics of PCF

Proposition. *For all typing judgements* $\Gamma \vdash M : \tau$, *the denotation*

$$[\![\Gamma \vdash M]\!] : [\![\Gamma]\!] \to [\![\tau]\!]$$

is a well-defined continous function.

Denotations of closed terms

For a closed term $M \in \mathrm{PCF}_\tau$, we get

$$[\![\emptyset \vdash M]\!] : [\![\emptyset]\!] \to [\![\tau]\!]$$

and, since $[\![\emptyset]\!] = \{ \bot \}$, we have

$$[\![M]\!] \stackrel{\text{def}}{=} [\![\emptyset \vdash M]\!](\bot) \in [\![\tau]\!] \qquad (M \in \mathrm{PCF}_\tau)$$

Compositionality

Proposition. *For all typing judgements* $\Gamma \vdash M : \tau$ *and* $\Gamma \vdash M' : \tau$, *and all contexts* $\mathcal{C}[-]$ *such that* $\Gamma' \vdash \mathcal{C}[M] : \tau'$ *and* $\Gamma' \vdash \mathcal{C}[M'] : \tau'$,

if $\llbracket \Gamma \vdash M \rrbracket = \llbracket \Gamma \vdash M' \rrbracket : \llbracket \Gamma \rrbracket \to \llbracket \tau \rrbracket$

then $\llbracket \Gamma' \vdash \mathcal{C}[M] \rrbracket = \llbracket \Gamma' \vdash \mathcal{C}[M] \rrbracket : \llbracket \Gamma' \rrbracket \to \llbracket \tau' \rrbracket$

Soundness

Proposition. *For all closed terms* $M, V \in \mathrm{PCF}_\tau$*,*

$$\text{if } M \Downarrow_\tau V \text{ then } [\![M]\!] = [\![V]\!] \in [\![\tau]\!] \;.$$

Substitution property

Proposition. *Suppose that* $\Gamma \vdash M : \tau$ *and that* $\Gamma[x \mapsto \tau] \vdash M' : \tau'$*, so that we also have* $\Gamma \vdash M'[M/x] : \tau'$*.*

Then,

$$\llbracket \Gamma \vdash M'[M/x] \rrbracket(\rho) = \llbracket \Gamma[x \mapsto \tau] \vdash M' \rrbracket \big(\rho\big[x \mapsto \llbracket \Gamma \vdash M \rrbracket(\rho)\big]\big)$$

for all $\rho \in \llbracket \Gamma \rrbracket$*.*

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for all $\rho \in \llbracket \Gamma \rrbracket$*.*

In particular when $\Gamma = \emptyset$, $\llbracket \langle x \mapsto \tau \rangle \vdash M' \rrbracket : \llbracket \tau \rrbracket \to \llbracket \tau' \rrbracket$ and

$$\llbracket M'[M/x] \rrbracket = \llbracket \langle x \mapsto \tau \rangle \vdash M' \rrbracket (\llbracket M \rrbracket)$$