

# Complexity Theory

## Lecture 7

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# Sets, Numbers and Scheduling

It is not just problems about formulas and graphs that turn out to be NP-complete.

Literally thousands of naturally arising problems have been proved NP-complete, in areas involving network design, scheduling, optimisation, data storage and retrieval, artificial intelligence and many others.

Such problems arise naturally whenever we have to construct a solution within constraints, and the most effective way appears to be an exhaustive search of an exponential solution space.

We now examine three more NP-complete problems, whose significance lies in that they have been used to prove a large number of other problems NP-complete, through reductions.

## 3D Matching

The decision problem of *3D Matching* is defined as:

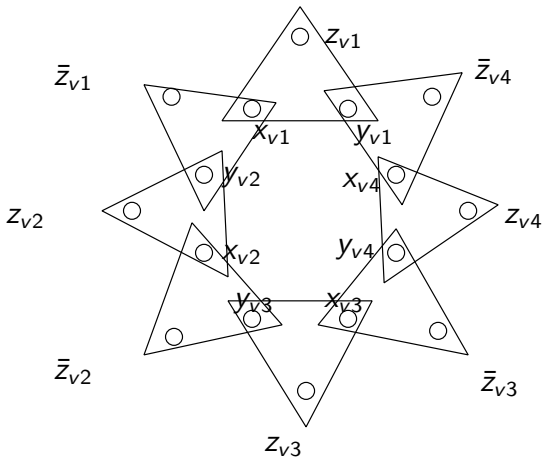
*Given three disjoint sets  $X$ ,  $Y$  and  $Z$ , and a set of triples  $M \subseteq X \times Y \times Z$ , does  $M$  contain a matching?*

*I.e. is there a subset  $M' \subseteq M$ , such that each element of  $X$ ,  $Y$  and  $Z$  appears in exactly one triple of  $M'$ ?*

We can show that **3DM** is **NP**-complete by a reduction from **3SAT**.

## Reduction

If a Boolean expression  $\phi$  in 3CNF has  $n$  variables, and  $m$  clauses, we construct for each variable  $v$  the following gadget.



In addition, for every clause  $c$ , we have two elements  $x_c$  and  $y_c$ .  
If the literal  $v$  occurs in  $c$ , we include the triple

$$(x_c, y_c, z_{vc})$$

in  $M$ .

Similarly, if  $\neg v$  occurs in  $c$ , we include the triple

$$(x_c, y_c, \bar{z}_{vc})$$

in  $M$ .

Finally, we include extra dummy elements in  $X$  and  $Y$  to make the numbers match up.

# Exact Set Covering

Two other well known problems are proved NP-complete by immediate reduction from 3DM.

*Exact Cover by 3-Sets* is defined by:

*Given a set  $U$  with  $3n$  elements, and a collection  $S = \{S_1, \dots, S_m\}$  of three-element subsets of  $U$ , is there a sub-collection containing exactly  $n$  of these sets whose union is all of  $U$ ?*

The reduction from 3DM simply takes  $U = X \cup Y \cup Z$ , and  $S$  to be the collection of three-element subsets resulting from  $M$ .

# Set Covering

More generally, we have the *Set Covering* problem:

*Given a set  $U$ , a collection  $S = \{S_1, \dots, S_m\}$  of subsets of  $U$  and an integer budget  $B$ , is there a collection of  $B$  sets in  $S$  whose union is  $U$ ?*

# Knapsack

**KNAPSACK** is a problem which generalises many natural scheduling and optimisation problems, and through reductions has been used to show many such problems **NP**-complete.

In the problem, we are given  $n$  items, each with a positive integer value  $v_i$  and weight  $w_i$ .

We are also given a maximum total weight  $W$ , and a minimum total value  $V$ .

*Can we select a subset of the items whose total weight does not exceed  $W$ , and whose total value is at least  $V$ ?*



# Reduction

The proof that **KNAPSACK** is **NP**-complete is by a reduction from the problem of **Exact Cover by 3-Sets**.

Given a set  $U = \{1, \dots, 3n\}$  and a collection of 3-element subsets of  $U$ ,  $S = \{S_1, \dots, S_m\}$ .

We map this to an instance of **KNAPSACK** with  $m$  elements each corresponding to one of the  $S_i$ , and having weight and value

$$\sum_{j \in S_i} (m+1)^{j-1}$$

and set the target weight and value both to

$$\sum_{j=0}^{3n-1} (m+1)^j$$

# Scheduling

Some examples of the kinds of scheduling tasks that have been proved NP-complete include:

## Timetable Design

*Given a set  $H$  of **work periods**, a set  $W$  of **workers** each with an associated subset of  $H$  (available periods), a set  $T$  of **tasks** and an assignment  $r : W \times T \rightarrow \mathbb{N}$  of **required work**, is there a mapping  $f : W \times T \times H \rightarrow \{0, 1\}$  which completes all tasks?*

# Scheduling

## Sequencing with Deadlines

Given a set  $T$  of *tasks* and for each task a *length*  $l \in \mathbb{N}$ , a release time  $r \in \mathbb{N}$  and a deadline  $d \in \mathbb{N}$ , is there a work schedule which completes each task between its release time and its deadline?

## Job Scheduling

Given a set  $T$  of *tasks*, a number  $m \in \mathbb{N}$  of processors a length  $l \in \mathbb{N}$  for each task, and an overall deadline  $D \in \mathbb{N}$ , is there a multi-processor schedule which completes all tasks by the deadline?