

Complexity Theory

Easter 2022

Suggested Exercises 2

1. Given a graph $G = (V, E)$, a set $U \subseteq V$ of vertices is called a *vertex cover* of G if, for each edge $(u, v) \in E$, either $u \in U$ or $v \in U$. That is, each edge has at least one end point in U . The decision problem **V-COVER** is defined as:

given a graph $G = (V, E)$, and an integer K , does G contain a vertex cover with K or *fewer* elements?

- (a) Show a polynomial time reduction from **IND** to **V-COVER**.
 - (b) Use (a) to argue that **V-COVER** is **NP**-complete.
2. The problem of four dimensional matching, **4DM**, is defined analogously with **3DM**:

Given four sets, W, X, Y and Z , each with n elements, and a set of quadruples $M \subseteq W \times X \times Y \times Z$, is there a subset $M' \subseteq M$, such that each element of W, X, Y and Z appears in exactly one tuple in M' .

Show that **4DM** is **NP**-complete.

3. Given a graph $G = (V, E)$, a *source vertex* $s \in V$ and a *target vertex* $t \in V$, a *Hamiltonian Path* from s to t in G is a path that begins at s , ends at t and visits every vertex in V exactly once. We define the decision problem **HamPath** as:

given a graph $G = (V, E)$ and vertices $s, t \in V$, does G contain a Hamiltonian path from s to t ?

- (a) Give a polynomial time reduction from the Hamiltonian cycle problem to **HamPath**.
- (b) Give a polynomial time reduction from **HamPath** to the problem of determining whether a graph has a Hamiltonian cycle.

Hint: consider adding a vertex to the graph.

4. We know from the Cook-Levin theorem that every problem in **NP** is reducible to **SAT**. Sometimes it is easy to give an explicit reduction. In this exercise you are asked to give such explicit reductions for two graph problems: **3-Col** and **HAM**. That is,

- (a) describe how to obtain, for any graph $G = (V, E)$, a Boolean expression ϕ_G so that ϕ_G is satisfiable if, and only if, G is 3-colourable; and
 - (b) describe how to obtain, for any graph $G = (V, E)$, a Boolean expression ϕ_G so that ϕ_G is satisfiable if, and only if, G contains a Hamiltonian cycle.
5. We use $x;0^n$ to denote the string that is obtained by concatenating the string x with a separator $;$ followed by n occurrences of 0. If $[M]$ represents the string encoding of a *non-deterministic* Turing machine M , show that the following language is NP-complete:

$$\{[M]; x; 0^n \mid M \text{ accepts } x \text{ within } n \text{ steps}\}.$$

Hint: rather than attempting a reduction from a particular NP-complete problem, it is easier to show this from first principles, i.e. construct a reduction for any NDTM M , and polynomial bound p .

6. **0-1 Integer Linear Programming.** An instance of a *linear programming* problem consists of a set $X = \{x_1, \dots, x_n\}$ of variables and a set of constraints, each of the form

$$\sum_{1 \leq i \leq n} c_i x_i \leq b,$$

where each c_i and b is an integer.

The 0-1 Integer Linear Programming Feasibility problem is, to determine, given such a linear programming problem, whether there is an assignment of values from the set $\{0, 1\}$ to the variables in X so that substituting these values in the constraints leads to all constraints being simultaneously satisfied.

Prove that this problem is NP-complete.

7. **Self-Reducibility.** *Self-reducibility* refers to the property of some problems in $L \in \text{NP}$, where the problem of finding a *witness* for the membership of an input x in L can be reduced to the decision problem for L . This question asks you to give such arguments in two specific instances.
- (a) Show that, given an oracle (i.e. a black box) for deciding whether a given graph $G = (V, E)$ is Hamiltonian, there is a polynomial-time algorithm that, on input G , outputs a Hamiltonian cycle in G if one exists.
 - (b) Show that, given an oracle for deciding whether a given graph G is 3-colourable, there is a polynomial-time algorithm that, on input G , produces a valid 3-colouring of G if one exists.