Lecture 16

Monads

Used in Haskell to abstract generic aspects of computation (return a value, sequencing) and to encapsulate effectful code.

Concept imported into functional programming from category theory, first for its denotational semantics by Moggi and then for its practice by Wadler.

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Here, a quick overview of:

- Moggi's computational λ-calculus and its categorical semantics using (strong) monads
- monads and adjunctions

Computational Lambda Calculus (CLC)

CLC extends STLC with new types, terms and equations...

Types: $A, B, \ldots ::=$ STLC types, plus

$\mathtt{T}(A)$ type of "computations" of values of type A

Terms: $s, t, \ldots ::=$ STLC terms, plus

$\mathtt{return}\, t$ trivial computation

$\mathtt{do}\{x \leftarrow s; t\}$ sequenced computation (binds free x in t)

As for STLC, we identify CLC syntax trees up to α-equivalence, where $=_\alpha$ is extended by the rules

$$\frac{t =_\alpha t'}{\mathtt{return}\, t =_\alpha \mathtt{return}\, t'} \quad \text{and} \quad \frac{s =_\alpha s' \qquad (y\ x) \cdot t =_\alpha (y\ x') \cdot t' \qquad y \text{ does not occur in } \{s, s', x, x', t, t'\}}{\mathtt{do}\{x \leftarrow s; t\} =_\alpha \mathtt{do}\{x' \leftarrow s'; t'\}}$$

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Typing rules:

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash \mathtt{return}\, t : \mathtt{T}(A)}\ (\textsc{val}) \qquad \frac{\Gamma \vdash s : \mathtt{T}(A) \qquad \Gamma, x : A \vdash t : \mathtt{T}(B)}{\Gamma \vdash \mathtt{do}\{x \leftarrow s; t\} : \mathtt{T}(B)}\ (\textsc{seq})$$

Equations…

CLC equations

Extend STLC $\beta\eta$-equality ($\Gamma \vdash s =_{\beta\eta} t : A$) to a relation $\Gamma \vdash s = t : A$ by adding the following rules:

$$\frac{\Gamma \vdash s : A \qquad \Gamma, x : A \vdash t : \mathtt{T}(B)}{\Gamma \vdash \mathtt{do}\{x \leftarrow \mathtt{return}\, s; t\} = t[s/x] : \mathtt{T}(B)}$$

$$\frac{\Gamma \vdash t : \mathtt{T}(A)}{\Gamma \vdash t = \mathtt{do}\{x \leftarrow t; \mathtt{return}\, x\} : \mathtt{T}(A)}$$

$$\frac{\Gamma \vdash s : \mathtt{T}(A) \qquad \Gamma, x : A \vdash t : \mathtt{T}(B) \qquad \Gamma, y : B \vdash u : \mathtt{T}(C)}{\Gamma \vdash \mathtt{do}\{y \leftarrow \mathtt{do}\{x \leftarrow s; t\}; u\} = \mathtt{do}\{x \leftarrow s; \mathtt{do}\{y \leftarrow t; u\}\}}$$

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$$\frac{\Gamma \vdash s : \mathtt{T}(A) \qquad \Gamma, x : A \vdash t : \mathtt{T}(B) \qquad \Gamma, y : B \vdash u : \mathtt{T}(C)}{\Gamma \vdash \mathtt{do}\{y \leftarrow \mathtt{do}\{x \leftarrow s; t\}; u\} = \mathtt{do}\{x \leftarrow s; \mathtt{do}\{y \leftarrow t; u\}\}}$$

(To describe a particular notion of computation (I/O, mutable state, exceptions, concurrent processes, ...) one can consider extensions of vanilla CLC, e.g. with extra ground types, constants and equations.)

Parameterised Kleisli triple

is the following extra structure on a category $\mathbf{C}$ with binary products:

- a function mapping each $X \in \texttt{obj}\,\mathbf{C}$ to an object $T(X) \in \texttt{obj}\,\mathbf{C}$
- for each $X \in \texttt{obj}\,\mathbf{C}$, a $\mathbf{C}$-morphism $X \xrightarrow{\eta_X} T(X)$
- for each $\mathbf{C}$-morphism $X \times Y \xrightarrow{f} T(Z)$ a $\mathbf{C}$-morphism $X \times T(Y) \xrightarrow{f^*} T(Z)$

satisfying…

Parameterised Kleisli triple[cont.]

- if $X \xrightarrow{f} X'$ and $X' \times Y \xrightarrow{g} T(Z)$, then

$$(g \circ (f \times \mathtt{id}_Y))^* = g^* \circ (f \times \mathtt{id}_{T(Y)})$$

- if $X \times Y \xrightarrow{f} T(Z)$, then

$$f^* \circ (\mathtt{id}_X \times \eta_Y) = f$$

- if $X \times Y \xrightarrow{f} T(Z)$ and $X \times Z \xrightarrow{g} T(W)$, then

$$(g^* \circ \langle \pi_1 \, , f \rangle)^* = g^* \circ \langle \pi_1 \, , f^* \rangle$$

Examples in **Set**

State: fix a set S (of "states") and define

$T(X) \triangleq (X \times S)^S$

$\eta_X \, x \, s \triangleq (x, s)$

$f^*(x, t) \, s \triangleq f(x, y) \, s'$ where $t \, s = (y, s')$

Examples in **Set**

State: fix a set S (of "states") and define

$$T(X) \triangleq (X \times S)^S$$

computations are functions $S \to X \times S$ taking states to values in X paired with a next state

$$\eta_X \, x \, s \triangleq (x, s)$$

$$f^*(x, t) \, s \triangleq f(x, y) \, s' \text{ where } t \, s = (y, s')$$

$f^*(x, \_)$ first "runs" $t \in T(Y)$ in state s to get (y, s'), then runs $f(x, y) \in T(Z)$ in the new state s'

Examples in **Set**

Error:

$$T(X) \triangleq X + 1 = \{(0, x) \mid x \in X\} \cup \{(1, 0)\}$$

$$\eta_X\, x \triangleq (0, x)$$

$$f^*(x, t) \triangleq \begin{cases} f(x, y) & \text{if } t = (0, y) \\ (1, 0) & \text{if } t = (1, 0) \end{cases}$$

Examples in **Set**

Error:

$$T(X) \triangleq X + 1 = \{(0, x) \mid x \in X\} \cup \{(1, 0)\}$$

computations are either copies $(0, x)$ of values in $x \in X$ or an error $(1, 0)$

$$\eta_X \, x \triangleq (0, x)$$

$$f^*(x, t) \triangleq \begin{cases} f(x, y) & \text{if } t = (0, y) \\ (1, 0) & \text{if } t = (1, 0) \end{cases}$$

if $t \in T(Y)$ is the error, then $f^*(x, \_)$ propagates it, otherwise it acts like f

Examples in **Set**

Continuations: fix a set R (of "results") and define

$$T(X) \triangleq R^{(R^X)}$$

$$\eta_X\, x \triangleq \lambda c \in R^X .\, c\, x$$

$$f^*(x, r) \triangleq \lambda c \in R^Z .\, r(\lambda y \in Y .\, f(x, y)\, c)$$

Examples in **Set**

Continuations: fix a set R (of "results") and define

$$T(X) \triangleq R^{(R^X)}$$

computations are functions $r : R^X \to R$ mapping continuations $c \in R^X$ of the computation to results $r\,c \in R$

$$\eta_X\, x \triangleq \lambda c \in R^X.\, c\, x$$

$$f^*(x, r) \triangleq \lambda c \in R^Z.\, r(\lambda y \in Y.\, f(x, y)\, c)$$

f^* maps a computation $r \in R^{(R^Y)}$ to the function taking a continuation $c \in R^Z$ to the result of applying r to the continuation $\lambda y \in Y.\, f(x, y)\, c$ in R^Y

Semantics of CLC

Given a ccc $\mathbf{C}$ equipped with a parameterised Kleisli triple $(T, \eta, (\_)^*)$, we can extend the semantics of STLC to one for CLC.

Computation types: $[\![\mathtt{T}(A)]\!] = T([\![A]\!])$

Trivial computations:

$$[\![\Gamma \vdash \mathtt{return}\, t : \mathtt{T}(A)]\!] = [\![\Gamma]\!] \xrightarrow{[\![\Gamma \vdash t:A]\!]} [\![A]\!] \xrightarrow{\eta_{[\![A]\!]}} T([\![A]\!])$$

Sequencing: $[\![\Gamma \vdash \mathtt{do}\{x \leftarrow s; t\} : \mathtt{T}(B)]\!] = f^* \circ \langle \mathtt{id}_{[\![\Gamma]\!]}, g\rangle$

where $\begin{cases} f &= [\![\Gamma]\!] \times [\![A]\!] \xrightarrow{[\![\Gamma, x:A \vdash t:\mathtt{T}(B)]\!]} T([\![B]\!]) \\ g &= [\![\Gamma]\!] \xrightarrow{[\![\Gamma \vdash s:\mathtt{T}(A)]\!]} T([\![A]\!]) \end{cases}$

(and where A is uniquely determined from the proof of $\Gamma \vdash \mathtt{do}\{x \leftarrow s; t\} : \mathtt{T}(B)$)

Semantics of CLC

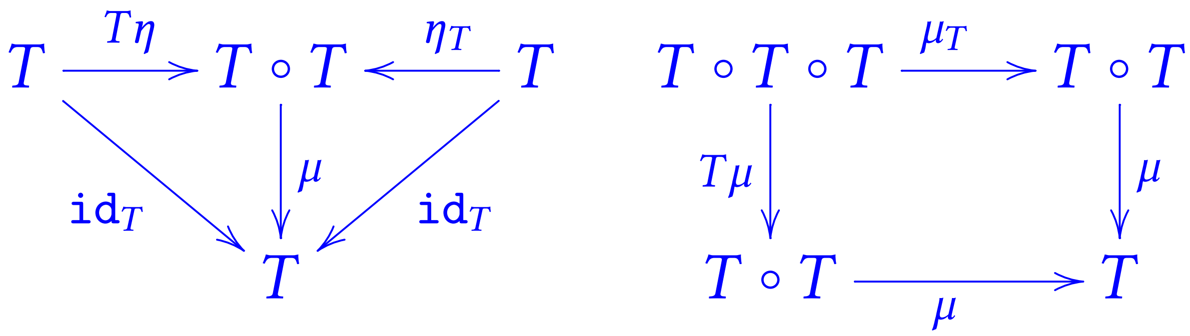
Given a ccc $\mathbf{C}$ equipped with a parameterised Kleisli triple $(T, \eta, (\_)^*)$, we can extend the semantics of STLC to one for CLC.

As for STLC versus cccs,

- the semantics of CLC in cc+Kleisli categories is equationally sound and complete
- one can use CLC as an internal language for describing constructs in cc+Kleisli categories
- there is a correspondence between equational theories in CLC and cc+Kleisli categories

Monads

A monad on a category $\mathbf{C}$ is given by a functor $T : \mathbf{C} \to \mathbf{C}$ and natural transformations $\eta : \mathtt{id} \to T$ and $\mu : T \circ T \to T$ satisfying



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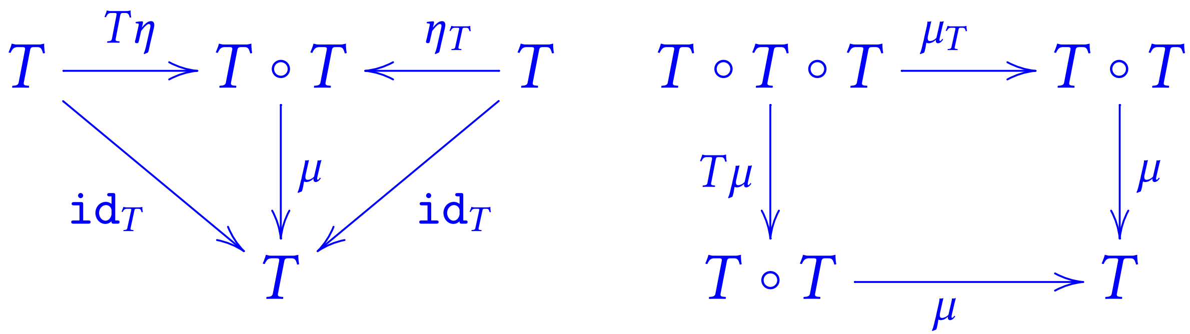

If $\mathbf{C}$ has binary products, then the monad is strong if there is a family of $\mathbf{C}$-morphisms $(X \times T(Y) \xrightarrow{s_{X,Y}} T(X \times Y) \mid X, Y \in \mathtt{obj}\,\mathbf{C})$ satisfying a number (7, in fact) of commutative diagrams (details omitted, see Moggi).

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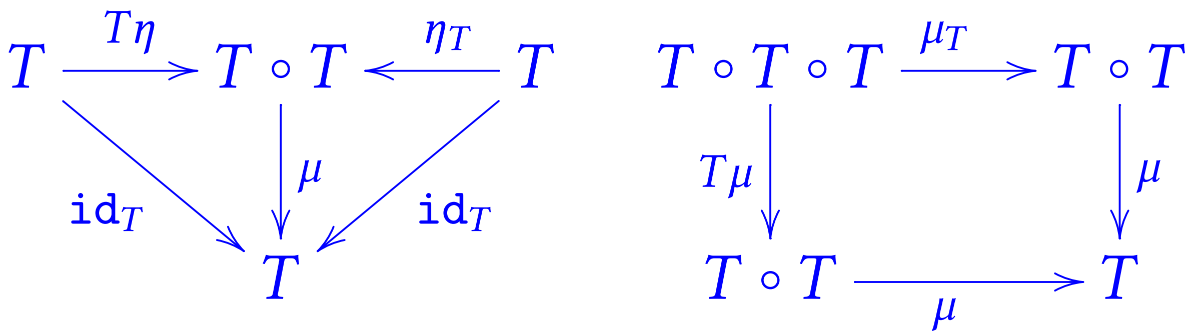

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FACT: for a given category with binary products, "parameterised Kleisli triple" and "strong monad" are equivalent notions – each gives rise to the other in a bijective fashion.

Monads and adjunctions

- Given an adjunction $\mathbf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{D}$ $\qquad F \dashv G$

  we get a monad $(G \circ F, \eta, \mu)$ on $\mathbf{C}$

  where $\begin{cases} \eta_X &= \overline{\mathtt{id}_{F\,X}} \\ \mu_X &= G(\overline{\mathtt{id}_{G(F\,X)}}) \end{cases}$

  E.g. for $\mathbf{Set} \underset{U}{\overset{F}{\rightleftarrows}} \mathbf{Mon}$ where U is the forgetful functor, $T = U \circ F$ is the list monad on $\mathbf{Set}$ ($T(X) = \mathtt{List}\,X$, η given by singleton lists, μ by flattening lists of lists). It's a strong monad (all monads of $\mathbf{Set}$ have a strength), but in general the monad associated with an adjunction may not be strong.

Monads and adjunctions

- Given an adjunction $\mathbf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{D}$ $\qquad F \dashv G$

  we get a monad $(G \circ F, \eta, \mu)$ on $\mathbf{C}$

- Given a monad (T, η, μ) on $\mathbf{C}$ we get an adjunction

  $\mathbf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{C}^T$ $\qquad F \dashv G$

Monads and adjunctions

- Given an adjunct

  we get a monad (

- Given a monad (

$$\mathbf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{C}^T \longleftarrow$$

$\mathbf{C}^T$ is the category of Eilenberg-Moore algebras for the monad T, which has objects (A, α) with $\alpha : T(A) \to A$ satisfying

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & T\,A \\ & {\scriptstyle \mathtt{id}_A} \searrow & \downarrow{\scriptstyle \alpha} \\ & & A \end{array} \qquad \begin{array}{ccc} T(T\,A) & \xrightarrow{\mu_A} & T\,A \\ {\scriptstyle T\,\alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ T\,A & \xrightarrow[\alpha]{} & A \end{array}$$

and morphisms $f\,(A, \alpha) \to (B, \beta)$ with $f : A \to B$ satisfying

$$\begin{array}{ccc} T\,A & \xrightarrow{T\,f} & T\,B \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta} \\ A & \xrightarrow[f]{} & B \end{array}$$

Monads and adjunctions

- Given an adjunction $\mathbf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{D}$ $\quad F \dashv G$

  we get a monad $(G \circ F, \eta, \mu)$ on $\mathbf{C}$

- Given a monad (T, η, μ) on $\mathbf{C}$ we get an adjunction

  $\mathbf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{C}^T$ $\quad F \dashv G$

- Starting from $\mathbf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{D}$ $\quad F \dashv G$ and forming the monad

  $T = G \circ F$, there's an obvious functor $K : \mathbf{D} \longrightarrow \mathbf{C}^T$.

  Monadicity Theorems impose conditions on $G : \mathbf{D} \rightarrow \mathbf{C}$ which ensure that K is an equivalence of categories. E.g. **Mon** is equivalent to the category of Eilenberg-Moore algebras for the list monad on **Set** (and similarly for any algebraic theory).

Some current themes involving category theory in computer science

- semantics of effects & co-effects in programming languages

  (monads and comonads)

- homotopy type theory

  (higher-dimensional category theory)

- structural aspects of networks, quantum computation/protocols, . . .

  (string diagrams for monoidal categories)