

Lecture 11

The category of small categories

Recall definition of **Cat**:

- ▶ objects are all small categories
- ▶ morphisms in **Cat(C, D)** are all functors $C \rightarrow D$
- ▶ composition and identity morphisms as for functors

Cat has a terminal object

The category

$$0 \begin{array}{c} \curvearrowright \\ \text{id}_0 \\ \curvearrowleft \end{array}$$

one object, one morphism

is terminal in **Cat**

Cat has binary products

Given small categories $\mathbf{C}, \mathbf{D} \in \mathbf{Cat}$, their product

$\mathbf{C} \xleftarrow{\pi_1} \mathbf{C} \times \mathbf{D} \xrightarrow{\pi_2} \mathbf{D}$ is:

Cat has binary products

Given small categories $\mathbf{C}, \mathbf{D} \in \mathbf{Cat}$, their product

$\mathbf{C} \xleftarrow{\pi_1} \mathbf{C} \times \mathbf{D} \xrightarrow{\pi_2} \mathbf{D}$ is:

- ▶ objects of $\mathbf{C} \times \mathbf{D}$ are pairs (X, Y) where $X \in \mathbf{C}$ and $Y \in \mathbf{D}$
- ▶ morphisms $(X, Y) \rightarrow (X', Y')$ in $\mathbf{C} \times \mathbf{D}$ are pairs (f, g) where $f \in \mathbf{C}(X, X')$ and $g \in \mathbf{D}(Y, Y')$
- ▶ composition and identity morphisms are given by those of \mathbf{C} (in the first component) and \mathbf{D} (in the second component)

$$\left\{ \begin{array}{l} \pi_1 \left((X, Y) \xrightarrow{(f, g)} (X', Y') \right) = X \xrightarrow{f} X' \\ \pi_2 \left((X, Y) \xrightarrow{(f, g)} (X', Y') \right) = Y \xrightarrow{g} Y' \end{array} \right.$$

Cat not only has finite products, it is also cartesian closed.

Exponentials in **Cat** are called **functor categories**.

To define them we need to consider **natural transformations**, which are the appropriate notion of morphism between functors.

Natural transformations

Motivating example: fix a set $S \in \mathbf{Set}$ and consider the two functors $F, G : \mathbf{Set} \rightarrow \mathbf{Set}$ given by

$$F \left(X \xrightarrow{f} Y \right) = S \times X \xrightarrow{\text{id}_S \times f} S \times Y$$
$$G \left(X \xrightarrow{f} Y \right) = X \times S \xrightarrow{f \times \text{id}_S} Y \times S$$

Natural transformations

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For each $X \in \mathbf{Set}$ there is an isomorphism (bijection) $\theta_X : FX \cong GX$ in \mathbf{Set} given by $\langle \pi_2, \pi_1 \rangle : S \times X \rightarrow X \times S$.

These isomorphisms do not depend on the particular nature of each set X (they are “polymorphic in X ”). One way to make this precise is...

...if we change from X to Y along a function $f : X \rightarrow Y$, then we get a commutative diagram in **Set**:

$$\begin{array}{ccc}
 S \times X & \xrightarrow{\langle \pi_2, \pi_1 \rangle} & X \times S \\
 \text{id} \times f \downarrow & & \downarrow f \times \text{id} \\
 S \times Y & \xrightarrow{\langle \pi_2, \pi_1 \rangle} & Y \times S
 \end{array}$$

The square commutes because for all $s \in S$ and $x \in X$

$$\begin{aligned}
 \langle \pi_2, \pi_1 \rangle((\text{id} \times f)(s, x)) &= \langle \pi_2, \pi_1 \rangle(s, f x) \\
 &= (f x, s) \\
 &= (f \times \text{id})(x, s) \\
 &= (f \times \text{id})(\langle \pi_2, \pi_1 \rangle(s, x))
 \end{aligned}$$

...if we change from X to Y along a function $f : X \rightarrow Y$, then we get a commutative diagram in **Set**:

$$\begin{array}{ccc} F X & \xrightarrow{\theta_X} & G X \\ F f \downarrow & & \downarrow G f \\ F Y & \xrightarrow{\theta_Y} & G Y \end{array}$$

We say that the family $(\theta_X \mid X \in \mathbf{Set})$ is **natural** in X .

Natural transformations

Definition. Given categories and functors $F, G : \mathbf{C} \rightarrow \mathbf{D}$, a **natural transformation** $\theta : F \rightarrow G$ is a family of \mathbf{D} -morphisms $\theta_X \in \mathbf{D}(F X, G X)$, one for each $X \in \mathbf{C}$, such that for all \mathbf{C} -morphisms $f : X \rightarrow Y$, the diagram

$$\begin{array}{ccc} F X & \xrightarrow{\theta_X} & G X \\ F f \downarrow & & \downarrow G f \\ F Y & \xrightarrow{\theta_Y} & G Y \end{array}$$

commutes in \mathbf{D} , that is, $\theta_Y \circ F f = G f \circ \theta_X$.

Example

Recall forgetful (U) and free (F) functors:

$$\mathbf{Set} \begin{array}{c} \xleftarrow{U} \\ \xrightarrow{F} \end{array} \mathbf{Mon}$$

There is a natural transformation $\eta : \text{id}_{\mathbf{Set}} \rightarrow U \circ F$, where for each $\Sigma \in \mathbf{Set}$

$$\eta_{\Sigma} : \Sigma \rightarrow U(F \Sigma) = \text{List } \Sigma$$

$$a \in \Sigma \mapsto [a] \in \text{List } \Sigma \text{ (one-element list)}$$

(Easy to see that

$$\begin{array}{ccc} \Sigma & \xrightarrow{\eta_{\Sigma}} & U(F \Sigma) \\ f \downarrow & & \downarrow U(F f) \\ \Sigma' & \xrightarrow{\eta_{\Sigma'}} & U(F \Sigma') \end{array} \text{ commutes.)}$$

Example

The **covariant powerset functor** $\mathcal{P} : \mathbf{Set} \rightarrow \mathbf{Set}$ is

$$\begin{aligned}\mathcal{P} X &\triangleq \{S \mid S \subseteq X\} \\ \mathcal{P} \left(X \xrightarrow{f} Y \right) &\triangleq \mathcal{P} X \xrightarrow{\mathcal{P} f} \mathcal{P} Y \\ S &\mapsto \mathcal{P} f S \triangleq \{f x \mid x \in S\}\end{aligned}$$

Example

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There is a natural transformation $\cup : \mathcal{P} \circ \mathcal{P} \rightarrow \mathcal{P}$ whose component at $X \in \mathbf{Set}$ sends $\mathcal{S} \in \mathcal{P}(\mathcal{P} X)$ to

$$\cup_X \mathcal{S} \triangleq \{x \in X \mid \exists S \in \mathcal{S}, x \in S\} \in \mathcal{P} X$$

(check that \cup_X is natural in X)

Non-example

The classic example of an “un-natural transformation” (the one that caused Eilenburg and MacLane to invent the concept of naturality) is the linear isomorphism between a finite dimensional real vectorspace V and its dual V^* (= vectorspace of linear functions $V \rightarrow \mathbb{R}$).

Both V and V^* have the same finite dimension, so are isomorphic by choosing bases; but there is no choice of basis for each V that makes the family of isomorphisms natural in V .

For a similar, more elementary non-example, see Ex. Sh. 5, question 4.

Composing natural transformations

Given functors $F, G, H : \mathbf{C} \rightarrow \mathbf{D}$ and natural transformations $\theta : F \rightarrow G$ and $\varphi : G \rightarrow H$, we get $\varphi \circ \theta : F \rightarrow H$ with

$$(\varphi \circ \theta)_X = \left(F X \xrightarrow{\theta_X} G X \xrightarrow{\varphi_X} H X \right)$$

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Check naturality:

$$\begin{aligned} H f \circ (\varphi \circ \theta)_X &\triangleq H f \circ \varphi_X \circ \theta_X && \text{naturality of } \varphi \\ &= \varphi_Y \circ G f \circ \theta_X && \text{naturality of } \theta \\ &= \varphi_Y \circ \theta_Y \circ F f \\ &\triangleq (\varphi \circ \theta)_Y \circ F f \end{aligned}$$

Identity natural transformation

Given a functor $F : \mathbf{C} \rightarrow \mathbf{D}$, we get a natural transformation $\text{id}_F : F \rightarrow F$ with

$$(\text{id}_F)_X = F X \xrightarrow{\text{id}_{FX}} F X$$

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Check naturality:

$$F f \circ (\text{id}_F)_X \triangleq F f \circ \text{id}_{FX} = F f = \text{id}_{FY} \circ F f \triangleq (\text{id}_F)_Y \circ F f$$

Functor categories

It is easy to see that composition and identities for natural transformations satisfy

$$(\psi \circ \varphi) \circ \theta = \psi \circ (\varphi \circ \theta)$$

$$\text{id}_G \circ \theta = \theta \circ \text{id}_F$$

so that we get a category:

Definition. Given categories \mathbf{C} and \mathbf{D} , the **functor category** $\mathbf{D}^{\mathbf{C}}$ has

- ▶ objects are all functors $\mathbf{C} \rightarrow \mathbf{D}$
- ▶ given $F, G : \mathbf{C} \rightarrow \mathbf{D}$, morphism from F to G in $\mathbf{D}^{\mathbf{C}}$ are the natural transformations $F \rightarrow G$
- ▶ composition and identity morphisms as above

If \mathcal{U} is a Grothendieck universe, then for each $X \in \mathcal{U}$ and $F \in \mathcal{U}^X$ we have that their **dependent product** and **dependent function** sets

$$\sum_{x \in X} F x \triangleq \{(x, y) \mid x \in X \wedge y \in F x\}$$

$$\prod_{x \in X} F x \triangleq \{f \subseteq \sum_{x \in X} F x \mid f \text{ is single-valued and total}\}$$

are also in \mathcal{U} ; and as a special case (of \prod , when F is a constant function with value Y) we also have that $X, Y \in \mathcal{U}$ implies $Y^X \in \mathcal{U}$.

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If \mathbf{C} and \mathbf{D} are small categories, then so is $\mathbf{D}^{\mathbf{C}}$.

because

$$\begin{aligned} \text{obj}(\mathbf{D}^{\mathbf{C}}) &\subseteq \sum_{F \in (\text{obj } \mathbf{D})^{\text{obj } \mathbf{C}}} \prod_{X, Y \in \text{obj } \mathbf{C}} \mathbf{D}(F X, F Y)^{\mathbf{C}(X, Y)} \\ \mathbf{D}^{\mathbf{C}}(F, G) &\subseteq \prod_{X \in \text{obj } \mathbf{C}} \mathbf{D}(F X, G X) \end{aligned}$$

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Aim to show that functor category $\mathbf{D}^{\mathbf{C}}$ is the exponential of \mathbf{C} and \mathbf{D} in \mathbf{Cat} ...

Cat is cartesian closed

Theorem. There is an **application functor**

$$\text{app} : \mathbf{D}^{\mathbf{C}} \times \mathbf{C} \rightarrow \mathbf{D}$$

that makes $\mathbf{D}^{\mathbf{C}}$ the exponential for \mathbf{C} and \mathbf{D} in **Cat**.

Given $(F, X) \in \mathbf{D}^{\mathbf{C}} \times \mathbf{C}$, we define

$$\text{app}(F, X) \triangleq F X$$

and given $(\theta, f) : (F, X) \rightarrow (G, Y)$ in $\mathbf{D}^{\mathbf{C}} \times \mathbf{C}$, we define

$$\begin{aligned} \text{app} \left((F, X) \xrightarrow{(\theta, f)} (G, Y) \right) &\triangleq F X \xrightarrow{F f} F Y \xrightarrow{\theta_Y} G Y \\ &= F X \xrightarrow{\theta_X} G X \xrightarrow{G f} G Y \end{aligned}$$

Check:
$$\begin{cases} \text{app}(\text{id}_F, \text{id}_X) &= \text{id}_{F X} \\ \text{app}(\varphi \circ \theta, g \circ f) &= \text{app}(\varphi, g) \circ \text{app}(\theta, f) \end{cases}$$

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that makes $\mathbf{D}^{\mathbf{C}}$ the exponential for \mathbf{C} and \mathbf{D} in **Cat**.

Definition of currying: given functor $F : \mathbf{E} \times \mathbf{C} \rightarrow \mathbf{D}$, we get a functor $\text{cur } F : \mathbf{E} \rightarrow \mathbf{D}^{\mathbf{C}}$ as follows. For each $Z \in \mathbf{E}$, $\text{cur } F Z \in \mathbf{D}^{\mathbf{C}}$ is the functor

$$\text{cur } F Z \left(\begin{array}{c} X \\ \downarrow f \\ X' \end{array} \right) \triangleq \begin{array}{c} F(Z, X) \\ \downarrow F(\text{id}_Z, f) \\ F(Z, X') \end{array}$$

For each $g : Z \rightarrow Z'$ in \mathbf{E} , $\text{cur } F g : \text{cur } F Z \rightarrow \text{cur } F Z'$ is the natural transformation whose component at each $X \in \mathbf{C}$ is

$$(\text{cur } F g)_X \triangleq F(g, \text{id}_X) : F(Z, X) \rightarrow F(Z', X)$$

(Check that this is natural in X ; and that $\text{cur } F$ preserves composition and identities in \mathbf{E} .)

Cat is cartesian closed

Theorem. There is an **application functor**

$$\text{app} : \mathbf{D}^{\mathbf{C}} \times \mathbf{C} \rightarrow \mathbf{D}$$

that makes $\mathbf{D}^{\mathbf{C}}$ the exponential for \mathbf{C} and \mathbf{D} in **Cat**.

Have to check that $\text{cur } F$ is the unique functor $G : \mathbf{E} \rightarrow \mathbf{D}^{\mathbf{C}}$ that makes

$$\begin{array}{ccc} \mathbf{E} \times \mathbf{C} & \xrightarrow{F} & \mathbf{D} \\ G \times \text{id}_{\mathbf{C}} \downarrow & & \nearrow \text{app} \\ \mathbf{D}^{\mathbf{C}} \times \mathbf{C} & & \end{array}$$

commute in **Cat** (exercise).