Lecture 9

STLC equations

take the form $\boxed{\Gamma \vdash s = t : A}$ where $\Gamma \vdash s : A$ and $\Gamma \vdash t : A$ are provable.

Such an equation is satisfied by the semantics in a ccc if $M[\![\Gamma \vdash s : A]\!]$ and $M[\![\Gamma \vdash t : A]\!]$ are equal $\mathbf{C}$-morphisms $M[\![\Gamma]\!] \longrightarrow M[\![A]\!]$.

Qu: which equations are always satisfied in any ccc?

Ans: $\beta\eta$-equivalence…

STLC $\beta\eta$-Equality

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- β-conversions

$$\boxed{\frac{\Gamma, x : A \vdash t : B \qquad \Gamma \vdash s : A}{\Gamma \vdash (\lambda x : A.\, t)s =_{\beta\eta} t[s/x] : B}} \quad \boxed{\frac{\Gamma \vdash s : A \qquad \Gamma \vdash t : B}{\Gamma \vdash \mathtt{fst}(s\,,\,t) =_{\beta\eta} s : A}}$$

$$\boxed{\frac{\Gamma \vdash s : A \qquad \Gamma \vdash t : B}{\Gamma \vdash \mathtt{snd}(s\,,\,t) =_{\beta\eta} t : B}}$$

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- η-conversions

$$\frac{\Gamma \vdash t : A \to B \qquad x \text{ does not occur in } t}{\Gamma \vdash t =_{\beta\eta} (\lambda x : A.\, t\, x) : A \to B}$$

$$\frac{\Gamma \vdash t : A \times B}{\Gamma \vdash t =_{\beta\eta} (\mathtt{fst}\, t\,,\, \mathtt{snd}\, t) : A \times B} \qquad \frac{\Gamma \vdash t : \mathtt{unit}}{\Gamma \vdash t =_{\beta\eta} () : \mathtt{unit}}$$

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- β-conversions
- η-conversions
- congruence rules

$$\boxed{\frac{\Gamma, x : A \vdash t =_{\beta\eta} t' : B}{\Gamma \vdash \lambda x : A.\, t =_{\beta\eta} \lambda x : A.\, t' : A \to B}}$$

$$\boxed{\frac{\Gamma \vdash s =_{\beta\eta} s' : A \to B \qquad \Gamma \vdash t =_{\beta\eta} t' : A}{\Gamma \vdash s\, t =_{\beta\eta} s'\, t' : B}}$$ etc

STLC $\beta\eta$-Equality

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- β-conversions
- η-conversions
- congruence rules
- $=_{\beta\eta}$ is reflexive, symmetric and transitive

$$\boxed{\frac{\Gamma \vdash t : A}{\Gamma \vdash t =_{\beta\eta} t : A}} \quad \boxed{\frac{\Gamma \vdash s =_{\beta\eta} t : A}{\Gamma \vdash t =_{\beta\eta} s : A}}$$

$$\boxed{\frac{\Gamma \vdash r =_{\beta\eta} s : A \qquad \Gamma \vdash s =_{\beta\eta} t : A}{\Gamma \vdash r =_{\beta\eta} t : A}}$$

STLC $\beta\eta$-Equality

Soundness Theorem for semantics of STLC in a ccc.
If $\Gamma \vdash s =_{\beta\eta} t : A$ is provable, then in any ccc

$$M[\![\Gamma \vdash s : A]\!] = M[\![\Gamma \vdash t : A]\!]$$

are equal $\mathbf{C}$-morphisms $M[\![\Gamma]\!] \to M[\![A]\!]$.

Proof is by induction on the structure of the proof of $\Gamma \vdash s =_{\beta\eta} t : A$.

Here we just check the case of β-conversion for functions.

So suppose we have $\Gamma, x : A \vdash t : B$ and $\Gamma \vdash s : A$. We have to see that

$$M[\![\Gamma \vdash (\lambda x : A.\, t)s : B]\!] = M[\![\Gamma \vdash t[s/x] : B]\!]$$

Suppose
$$
\begin{aligned}
M[\![\Gamma]\!] &= X \\
M[\![A]\!] &= Y \\
M[\![B]\!] &= Z \\
M[\![\Gamma, x : A \vdash t : B]\!] &= f : X \times Y \longrightarrow Z \\
M[\![\Gamma \vdash s : A]\!] &= g : X \longrightarrow Z
\end{aligned}
$$

Then
$$M[\![\Gamma \vdash \lambda x : A.\, t : A \to B]\!] = \mathtt{cur}\, f : X \longrightarrow Z^Y$$

and hence

$$
\begin{aligned}
& M[\![\Gamma \vdash (\lambda x : A.\, t)s : B]\!] \\
& = \mathtt{app} \circ \langle \mathtt{cur}\, f \,,\, g \rangle \\
& = \mathtt{app} \circ (\mathtt{cur}\, f \times \mathtt{id}_Y) \circ \langle \mathtt{id}_X \,,\, g \rangle && \text{since } (a \times b) \circ \langle c \,,\, d \rangle = \langle a \circ c \,,\, b \circ d \rangle \\
& = f \circ \langle \mathtt{id}_X \,,\, g \rangle && \text{by definition of } \mathtt{cur}\, f \\
& = M[\![\Gamma \vdash t[s/x] : B]\!] && \text{by the Substitution Theorem}
\end{aligned}
$$

as required.

The internal language of a ccc, $\mathbf{C}$

- one ground type for each $\mathbf{C}$-object X
- for each $X \in \mathbf{C}$, one constant f^X for each $\mathbf{C}$-morphism $f : 1 \to X$ ("global element" of the object X)

The types and terms of STLC over this language usefully describe constructions on the objects and morphisms of $\mathbf{C}$ using its cartesian closed structure, but in an "element-theoretic" way.

For example…

Example

In any ccc $\mathbf{C}$, for any $X, Y, Z \in \mathbf{C}$ there is an isomorphism

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In any ccc $\mathbf{C}$, for any $X, Y, Z \in \mathbf{C}$ there is an isomorphism

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which in the internal language of $\mathbf{C}$ is described by the terms

$$\diamond \vdash s : ((X \times Y) \to Z) \to (X \to (Y \to Z))$$
$$\diamond \vdash t : (X \to (Y \to Z)) \to ((X \times Y) \to Z)$$

where $\begin{cases} s & \triangleq \lambda f : (X \times Y) \to Z .\, \lambda x : X .\, \lambda y : Y .\, f(x\,, y) \\ t & \triangleq \lambda g : X \to (Y \to Z) .\, \lambda z : X \times Y .\, g\,(\mathtt{fst}\, z)\,(\mathtt{snd}\, z) \end{cases}$ and

which satisfy $\begin{cases} \diamond, f : (X \times Y) \to Z \vdash t(s\, f) =_{\beta\eta} f \\ \diamond, g : X \to (Y \to Z) \vdash s(t\, g) =_{\beta\eta} g \end{cases}$

Free cartesian closed categories

The Soundness Theorem has a converse—completeness.

In fact for a given set of ground types and typed constants there is a single ccc $\boxed{\mathbf{F}}$ (the free ccc for that language) with an interpretation function M so that $\Gamma \vdash s =_{\beta\eta} t : A$ is provable iff $M[\![\Gamma \vdash s : A]\!] = M[\![\Gamma \vdash t : A]\!]$ in $\mathbf{F}$.

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- $\mathbf{F}$-objects are the STLC types over the given set of ground types
- $\mathbf{F}$-morphisms $A \longrightarrow B$ are equivalence classes of STLC terms t satisfying $\diamond \vdash t : A \to B$ (so t is a *closed* term—it has no free variables) with respect to the equivalence relation equating s and t if $\diamond \vdash s =_{\beta\eta} t : A \to B$ is provable.
- identity morphism on A is the equivalence class of $\diamond \vdash \lambda x : A.\, x : A \to A$.
- composition of a morphism $A \longrightarrow B$ represented by $\diamond \vdash s : A \to B$ and a morphism $B \longrightarrow C$ represented by $\diamond \vdash t : B \to C$ is represented by $\diamond \vdash \lambda x : A.\, t(s\,x) : A \to C$.

Curry-Howard correspondence

Logic		**Type Theory**
propositions	$\leftrightarrow$	types
proofs	$\leftrightarrow$	terms

E.g. IPL *versus* STLC.

Curry-Howard for IPL *vs* STLC

Proof of $\diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta$ in IPL

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\cdots}\,(\text{AX})}{\Phi \vdash \psi \Rightarrow \theta}\,(\text{WK}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\cdots}\,(\text{AX})}{\cdots}\,(\text{WK})}{\Phi \vdash \varphi \Rightarrow \psi}\,(\text{WK}) \qquad \dfrac{}{\Phi \vdash \varphi}\,(\text{AX})}{\Phi \vdash \psi}\,(\Rightarrow\text{E})}{\Phi \vdash \theta}\,(\Rightarrow\text{E})}{\diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta}\,(\Rightarrow\text{I})$$

where $\Phi = \diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta, \varphi$

Curry-Howard for IPL *vs* STLC

and a corresponding STLC term

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\cdots}\,(\textsc{ax})}{\Phi \vdash z : \psi \Rightarrow \theta}\,(\textsc{wk}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\cdots}\,(\textsc{ax})}{\cdots}\,(\textsc{wk})}{\Phi \vdash y : \varphi \Rightarrow \psi}\,(\textsc{wk}) \quad \dfrac{}{\Phi \vdash x : \varphi}\,(\textsc{ax})}{\Phi \vdash y\,x : \psi}\,({\Rightarrow}\textsc{e})}{\Phi \vdash z(y\,x) : \theta}\,({\Rightarrow}\textsc{e})}{\diamond, y : \varphi \Rightarrow \psi, z : \psi \Rightarrow \theta \vdash \lambda x : \varphi.\, z(y\,x) : \varphi \Rightarrow \theta}\,({\Rightarrow}\textsc{i})$$

where $\Phi = \diamond, y : \varphi \Rightarrow \psi, z : \psi \Rightarrow \theta, x : \varphi$

Curry-Howard-Lawvere/Lambek correspondence

Logic		**Type Theory**		**Category Theory**
propositions	$\leftrightarrow$	types	$\leftrightarrow$	objects
proofs	$\leftrightarrow$	terms	$\leftrightarrow$	morphisms

E.g. IPL *versus* STLC *versus* CCCs

Curry-Howard-Lawvere/Lambek correspondence

Logic		**Type Theory**		**Category Theory**
propositions	$\leftrightarrow$	types	$\leftrightarrow$	objects
proofs	$\leftrightarrow$	terms	$\leftrightarrow$	morphisms

E.g. IPL *versus* STLC *versus* CCCs

These correspondences can be made into category-theoretic equivalences—we first need to define the notions of functor and natural transformation in order to define the notion of equivalence of categories.