Lecture 9

STLC equations

take the form $\Gamma \vdash s = t : A$ where $\Gamma \vdash s : A$ and $\Gamma \vdash t : A$ are provable.

Such an equation is satisfied by the semantics in a ccc if $M[\Gamma \vdash s : A]$ and $M[\Gamma \vdash t : A]$ are equal C-morphisms $M[\Gamma] \rightarrow M[A]$.

Qu: which equations are always satisfied in any ccc? Ans: $\beta\eta$ -equivalence...

The relation $\Gamma \vdash s =_{\beta\eta} t : A$ (where Γ ranges over typing environments, *s* and *t* over terms and *A* over types) is inductively defined by the following rules:

β -conversions

 $\begin{array}{|c|c|c|c|c|}\hline \Gamma, x : A \vdash t : B & \Gamma \vdash s : A \\\hline \Gamma \vdash (\lambda x : A. t)s =_{\beta\eta} t[s/x] : B \\\hline \hline \Gamma \vdash fst(s, t) =_{\beta\eta} s : A \\\hline \hline \Gamma \vdash s : A & \Gamma \vdash t : B \\\hline \Gamma \vdash snd(s, t) =_{\beta\eta} t : B \\\hline \end{array}$

•
$$\beta$$
-conversions

$$\begin{array}{l} \hline \Gamma \vdash t : A \rightarrow B & x \text{ does not occur in } t \\ \hline \Gamma \vdash t =_{\beta\eta} (\lambda x : A. t x) : A \rightarrow B \end{array} \\ \hline \\ \hline \Gamma \vdash t : A \times B & \Gamma \vdash t : \text{ unit} \\ \hline \Gamma \vdash t =_{\beta\eta} (\text{fst } t, \text{ snd } t) : A \times B & \Gamma \vdash t =_{\beta\eta} () : \text{ unit} \end{array}$$

- \blacktriangleright β -conversions
- \blacktriangleright η -conversions
- congruence rules

$$\Gamma, x : A \vdash t =_{\beta\eta} t' : B$$

$$\Gamma \vdash \lambda x : A. t =_{\beta\eta} \lambda x : A. t' : A \to B$$

$$\Gamma \vdash s =_{\beta\eta} s' : A \to B \qquad \Gamma \vdash t =_{\beta\eta} t' : A$$

$$\Gamma \vdash s t =_{\beta\eta} s' t' : B$$
etc

- β -conversions
- \blacktriangleright η -conversions
- congruence rules

$$=_{\beta\eta} \text{ is reflexive, symmetric and transitive}$$

$$\boxed{\Gamma \vdash t : A} \qquad \boxed{\Gamma \vdash s =_{\beta\eta} t : A} \qquad \boxed{\Gamma \vdash s =_{\beta\eta} t : A} \qquad \boxed{\Gamma \vdash t =_{\beta\eta} s : A} \qquad \boxed{\Gamma \vdash r =_{\beta\eta} s : A} \qquad \boxed{\Gamma \vdash r =_{\beta\eta} t : A} \qquad \boxed{\Gamma \vdash r =$$

Soundness Theorem for semantics of STLC in a ccc. If $\Gamma \vdash s =_{\beta\eta} t : A$ is provable, then in any ccc

 $M[\Gamma \vdash s : A] = M[\Gamma \vdash t : A]$

are equal C-morphisms $M[[\Gamma]] \rightarrow M[[A]]$.

Proof is by induction on the structure of the proof of $\Gamma \vdash s =_{\beta\eta} t : A$.

Here we just check the case of β -conversion for functions.

So suppose we have Γ , $x : A \vdash t : B$ and $\Gamma \vdash s : A$. We have to see that

 $M[\![\Gamma \vdash (\lambda x : A. t)s : B]\!] = M[\![\Gamma \vdash t[s/x] : B]\!]$

Suppose
$$M[[\Gamma]] = X$$

 $M[[A]] = Y$
 $M[[B]] = Z$
 $M[[\Gamma, x : A \vdash t : B]] = f : X \times Y \rightarrow Z$
 $M[[\Gamma \vdash s : A]] = g : X \rightarrow Z$

Then

$$M[\![\Gamma \vdash \lambda x : A. t : A \to B]\!] = \operatorname{cur} f : X \to Z^Y$$

and hence

$$M[\Gamma \vdash (\lambda x : A. t)s : B]]$$

= app \cur f, g
= app \cur f \times id_Y) \circ \langle id_X, g
= f \circ \langle id_X, g
= M[[\Gamma \dot t[s/x]] : B]]

as required.

since $(a \times b) \circ \langle c, d \rangle = \langle a \circ c, b \circ d \rangle$ by definition of cur fby the <u>Substitution Theorem</u>

The internal language of a ccc, C

- one ground type for each C-object X
- ► for each $X \in \mathbb{C}$, one constant f^X for each C-morphism $f : 1 \rightarrow X$ ("global element" of the object X)

The types and terms of STLC over this language usefully describe constructions on the objects and morphisms of C using its cartesian closed structure, but in an "element-theoretic" way.

For example...

Example

In any ccc C, for any $X, Y, Z \in C$ there is an isomorphism

 $Z^{(X \times Y)} \cong (Z^Y)^X$

Example

In any ccc **C**, for any $X, Y, Z \in \mathbf{C}$ there is an isomorphism $Z^{(X \times Y)} \cong (Z^Y)^X$

which in the internal language of **C** is described by the terms

$$\diamond \vdash s : ((X \times Y) \to Z) \to (X \to (Y \to Z))$$
$$\diamond \vdash t : (X \to (Y \to Z)) \to ((X \times Y) \to Z)$$

where
$$\begin{cases} s &\triangleq \lambda f : (X \times Y) \to Z. \ \lambda x : X. \ \lambda y : Y. \ f(x, y) \\ t &\triangleq \lambda g : X \to (Y \to Z). \ \lambda z : X \times Y. \ g \ (\texttt{fst} z) \ (\texttt{snd} z) \end{cases} \text{ and} \\ \text{which satisfy} \begin{cases} \diamond, f : (X \times Y) \to Z \vdash t(s f) =_{\beta\eta} f \\ \diamond, g : X \to (Y \to Z) \vdash s(t g) =_{\beta\eta} g \end{cases}$$

Free cartesian closed categories

The Soundness Theorem has a converse-completeness.

In fact for a given set of ground types and typed constants there is a single ccc **F** (the free ccc for that language) with an interpretation function *M* so that $\Gamma \vdash s =_{\beta\eta} t : A$ is provable iff $M[\Gamma \vdash s : A] = M[\Gamma \vdash t : A]$ in **F**.

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- F-objects are the STLC types over the given set of ground types
- F-morphisms $A \to B$ are equivalence classes of STLC terms t satisfying $\diamond \vdash t : A \to B$ (so t is a *closed* term—it has no free variables) with respect to the equivalence relation equating s and t if $\diamond \vdash s =_{\beta\eta} t : A \to B$ is provable.
- identity morphism on A is the equivalence class of $\diamond \vdash \lambda x : A \cdot x : A \rightarrow A$.
- composition of a morphism $A \to B$ represented by $\diamond \vdash s : A \to B$ and a morphism $B \to C$ represented by $\diamond \vdash t : B \to C$ is represented by $\diamond \vdash \lambda x : A. t(s x) : A \to C$.

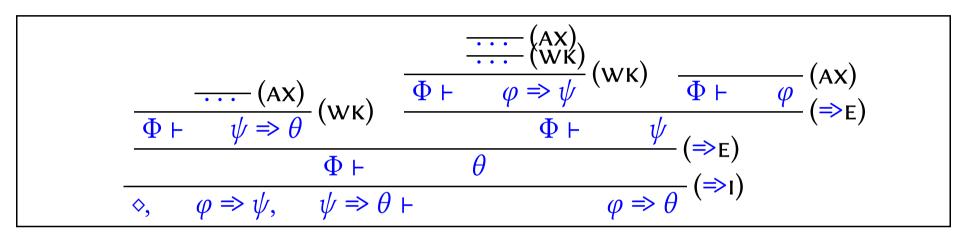
Curry-Howard correspondence

Logic		Theory	
propositions	\leftrightarrow	types	
proofs	\leftrightarrow	terms	

E.g. IPL versus STLC.

Curry-Howard for IPL vs STLC

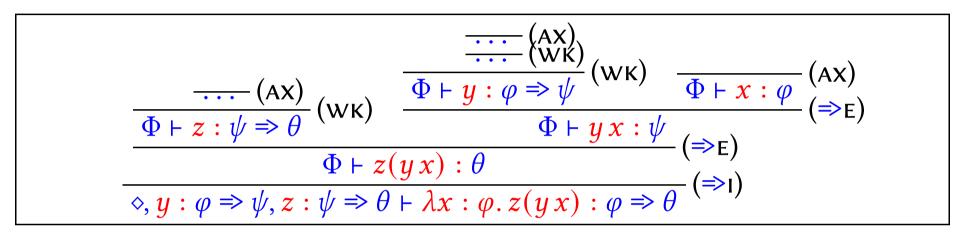
Proof of $\diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta$ in IPL



where $\Phi = \diamond$, $\varphi \Rightarrow \psi$, $\psi \Rightarrow \theta$, φ

Curry-Howard for IPL vs STLC

and a corresponding STLC term



where $\Phi = \diamond, y : \varphi \Rightarrow \psi, z : \psi \Rightarrow \theta, x : \varphi$

Curry-Howard-Lawvere/Lambek correspondence

Logic	Type Theory		Category Theory	
propositions	\leftrightarrow	types	\leftrightarrow	objects
proofs	\leftrightarrow	terms	\leftrightarrow	morphisms

E.g. IPL versus STLC versus CCCs

Curry-Howard-Lawvere/Lambek correspondence

		Туре		Category
Logic		Theory		Theory
propositions	\leftrightarrow	types	\leftrightarrow	objects
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E.g. IPL versus STLC versus CCCs

These correspondences can be made into category-theoretic equivalences—we first need to define the notions of functor and natural transformation in order to define the notion of equivalence of categories.