Category Theory

Lecture 7

- Exercise Sheet 3 – available on course web page
- Solution notes for Ex. Sh. 2 – available on Moodle

IPL entailment $\Phi \vdash \varphi$

Recall the rules:

$$\frac{}{\Phi, \varphi \vdash \varphi}\ (\textsc{ax}) \qquad \frac{\Phi \vdash \varphi}{\Phi, \psi \vdash \varphi}\ (\textsc{wk}) \qquad \frac{\Phi \vdash \varphi \qquad \Phi, \varphi \vdash \psi}{\Phi \vdash \psi}\ (\textsc{cut})$$

$$\frac{}{\Phi \vdash \mathtt{true}}\ (\textsc{true}) \qquad \frac{\Phi \vdash \varphi \qquad \Phi \vdash \psi}{\Phi \vdash \varphi \,\&\, \psi}\ (\&\textsc{i}) \qquad \frac{\Phi, \varphi \vdash \psi}{\Phi \vdash \varphi \Rightarrow \psi}\ (\Rightarrow\textsc{i})$$

$$\frac{\Phi \vdash \varphi \,\&\, \psi}{\Phi \vdash \varphi}\ (\&\textsc{e}_1) \qquad \frac{\Phi \vdash \varphi \,\&\, \psi}{\Phi \vdash \psi}\ (\&\textsc{e}_2) \qquad \frac{\Phi \vdash \varphi \Rightarrow \psi \qquad \Phi \vdash \varphi}{\Phi \vdash \psi}\ (\Rightarrow\textsc{e})$$

Proof theory

Two IPL proofs of $\diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta$

$$
\dfrac{
  \dfrac{\dfrac{}{\cdots}\,(\textsc{ax})}{\Phi, \varphi \vdash \psi \Rightarrow \theta}\,(\textsc{wk})
  \qquad
  \dfrac{
    \dfrac{\dfrac{\dfrac{}{\cdots}\,(\textsc{ax})}{\cdots}\,(\textsc{wk})}{\Phi, \varphi \vdash \varphi \Rightarrow \psi}\,(\textsc{wk})
    \qquad
    \dfrac{}{\Phi, \varphi \vdash \varphi}\,(\textsc{ax})
  }{\Phi, \varphi \vdash \psi}\,(\Rightarrow\textsc{e})
}{
  \dfrac{\Phi, \varphi \vdash \theta}{\Phi \vdash \varphi \Rightarrow \theta}\,(\Rightarrow\textsc{i})
}\,(\Rightarrow\textsc{e})
$$

where $\Phi \triangleq \diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta$

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{\dfrac{\dfrac{}{\cdots}\,(\textsc{ax})}{\cdots}\,(\textsc{wk})}{\Psi \vdash \varphi \Rightarrow \psi}\,(\textsc{wk})
      \qquad
      \dfrac{}{\Psi \vdash \varphi}\,(\textsc{ax})
    }{\Psi \vdash \psi}\,(\Rightarrow\textsc{e})
    \qquad
    \dfrac{
      \dfrac{\dfrac{\dfrac{}{\cdots}\,(\textsc{ax})}{\cdots}\,(\textsc{wk})}{\Psi, \psi \vdash \psi \Rightarrow \theta}\,(\textsc{wk})
      \qquad
      \dfrac{}{\Psi, \psi \vdash \psi}\,(\textsc{ax})
    }{\Psi, \psi \vdash \theta}\,(\Rightarrow\textsc{e})
  }{\Psi \vdash \theta}\,(\textsc{cut})
}{\diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta}\,(\Rightarrow\textsc{i})
$$

where $\Psi \triangleq \diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta, \varphi$

Proof theory

Two IPL proofs of $\diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\cdots}\,(\textsc{ax})}{\Phi, \varphi \vdash \psi \Rightarrow \theta}\,(\textsc{wk}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\cdots}\,(\textsc{ax})}{\cdots}\,(\textsc{wk})}{\Phi, \varphi \vdash \varphi \Rightarrow \psi}\,(\textsc{wk}) \qquad \dfrac{}{\Phi, \varphi \vdash \varphi}\,(\textsc{ax})}{\Phi, \varphi \vdash \psi}\,(\Rightarrow\textsc{e})}{\dfrac{\Phi, \varphi \vdash \theta}{\Phi \vdash \varphi \Rightarrow \theta}\,(\Rightarrow\textsc{i})}\,(\Rightarrow\textsc{e})$$

where $\Phi \triangleq \diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\cdots}\,(\textsc{ax})}{\cdots}\,(\textsc{wk})}{\Psi \vdash \varphi \Rightarrow \psi}\,(\textsc{wk}) \qquad \dfrac{}{\Psi \vdash \varphi}\,(\textsc{ax})}{\Psi \vdash \psi}\,(\Rightarrow\textsc{e}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\cdots}\,(\textsc{ax})}{\cdots}\,(\textsc{wk})}{\Psi, \psi \vdash \psi \Rightarrow \theta}\,(\textsc{wk}) \qquad \dfrac{}{\Psi, \psi \vdash \psi}\,(\textsc{ax})}{\Psi, \psi \vdash \theta}\,(\Rightarrow\textsc{e})}{\Psi \vdash \theta}\,(\textsc{cut})}{\diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta}\,(\Rightarrow\textsc{i})$$

where $\Psi \triangleq \diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta, \varphi$

Why is the first proof simpler than the second one?

Proof theory

$$\frac{}{\Phi, \varphi \vdash \varphi}\ (\textsc{ax}) \qquad \frac{\Phi \vdash \varphi}{\Phi, \psi \vdash \varphi}\ (\textsc{wk}) \qquad \frac{\Phi \vdash \varphi \qquad \Phi, \varphi \vdash \psi}{\Phi \vdash \psi}\ (\textsc{cut})$$

$$\frac{}{\Phi \vdash \texttt{true}}\ (\textsc{true}) \qquad \frac{\Phi \vdash \varphi \qquad \Phi \vdash \psi}{\Phi \vdash \varphi \,\&\, \psi}\ (\&\textsc{i}) \qquad \frac{\Phi, \varphi \vdash \psi}{\Phi \vdash \varphi \Rightarrow \psi}\ ({\Rightarrow}\textsc{i})$$

$$\frac{\Phi \vdash \varphi \,\&\, \psi}{\Phi \vdash \varphi}\ (\&\textsc{e}_1) \qquad \frac{\Phi \vdash \varphi \,\&\, \psi}{\Phi \vdash \psi}\ (\&\textsc{e}_2) \qquad \frac{\Phi \vdash \varphi \Rightarrow \psi \qquad \Phi \vdash \varphi}{\Phi \vdash \psi}\ ({\Rightarrow}\textsc{e})$$

FACT: if an IPL sequent $\Phi \vdash \phi$ is provable from the rules, it is provable without using the (CUT) rule.

Proof theory

$$\frac{}{\Phi, \varphi \vdash \varphi}\ (\textsc{ax}) \qquad \frac{\Phi \vdash \varphi}{\Phi, \psi \vdash \varphi}\ (\textsc{wk}) \qquad \frac{\Phi \vdash \varphi \qquad \Phi, \varphi \vdash \psi}{\Phi \vdash \psi}\ (\textsc{cut})$$

$$\frac{}{\Phi \vdash \mathtt{true}}\ (\textsc{true}) \qquad \frac{\Phi \vdash \varphi \qquad \Phi \vdash \psi}{\Phi \vdash \varphi \,\&\, \psi}\ (\&\textsc{i}) \qquad \frac{\Phi, \varphi \vdash \psi}{\Phi \vdash \varphi \Rightarrow \psi}\ ({\Rightarrow}\textsc{i})$$

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FACT: if an IPL sequent $\Phi \vdash \phi$ is provable from the rules, it is provable without using the (CUT) rule.

Simply-Typed Lambda Calculus provides a language for describing proofs in IPL and their properties…

Proof theory

Two IPL proofs of $\diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\cdots}\,\text{(AX)}}{\Phi, \varphi \vdash \psi \Rightarrow \theta}\,\text{(WK)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\cdots}\,\text{(AX)}}{\cdots}\,\text{(WK)}}{\Phi, \varphi \vdash \varphi \Rightarrow \psi}\,\text{(WK)} \quad \dfrac{}{\Phi, \varphi \vdash \varphi}\,\text{(AX)}}{\Phi, \varphi \vdash \psi}\,(\Rightarrow\text{E})}{\Phi, \varphi \vdash \theta}\,(\Rightarrow\text{E})}{\Phi \vdash \varphi \Rightarrow \theta}\,(\Rightarrow\text{I})$$

where $\Phi \triangleq \diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\cdots}\,\text{(AX)}}{\cdots}\,\text{(WK)}}{\Psi \vdash \varphi \Rightarrow \psi}\,\text{(WK)} \quad \dfrac{}{\Psi \vdash \varphi}\,\text{(AX)}}{\Psi \vdash \psi}\,(\Rightarrow\text{E}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\cdots}\,\text{(AX)}}{\cdots}\,\text{(WK)}}{\Psi, \psi \vdash \psi \Rightarrow \theta}\,\text{(WK)} \quad \dfrac{}{\Psi, \psi \vdash \psi}\,\text{(AX)}}{\Psi, \psi \vdash \theta}\,(\Rightarrow\text{E})}{\Psi \vdash \theta}\,\text{(CUT)}}{\diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta}\,(\Rightarrow\text{I})$$

where $\Psi \triangleq \diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta, \varphi$

$$\diamond, f : \varphi \Rightarrow \psi, g : \psi \Rightarrow \theta \vdash \lambda x : \varphi.\ (\lambda y : \psi.\ g\,y)(f\,x) \ : \varphi \Rightarrow \theta$$

Proof theory

Two IPL proofs of $\diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\cdots}\,(\text{AX})}{\Phi, \varphi \vdash \psi \Rightarrow \theta}\,(\text{WK}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\cdots}\,(\text{AX})}{\cdots}\,(\text{WK})}{\Phi, \varphi \vdash \varphi \Rightarrow \psi}\,(\text{WK}) \quad \dfrac{}{\Phi, \varphi \vdash \varphi}\,(\text{AX})}{\Phi, \varphi \vdash \psi}\,(\Rightarrow\text{E})}{\Phi, \varphi \vdash \theta}\,(\Rightarrow\text{E})}{\Phi \vdash \varphi \Rightarrow \theta}\,(\Rightarrow\text{I})$$

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$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\cdots}\,(\text{AX})}{\cdots}\,(\text{WK})}{\Psi \vdash \varphi \Rightarrow \psi}\,(\text{WK}) \quad \dfrac{}{\Psi \vdash \varphi}\,(\text{AX})}{\Psi \vdash \psi}\,(\Rightarrow\text{E}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\cdots}\,(\text{AX})}{\cdots}\,(\text{WK})}{\Psi, \psi \vdash \psi \Rightarrow \theta}\,(\text{WK}) \quad \dfrac{}{\Psi, \psi \vdash \psi}\,(\text{AX})}{\Psi, \psi \vdash \theta}\,(\Rightarrow\text{E})}{\Psi \vdash \theta}\,(\text{CUT})}{\diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta}\,(\Rightarrow\text{I})$$

where $\Psi \triangleq \diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta, \varphi$

$\diamond, f : \varphi \Rightarrow \psi, g : \psi \Rightarrow \theta \vdash \lambda x : \varphi.\ (\lambda y : \psi.\ g\, y)(f\, x) \ : \varphi \Rightarrow \theta$

$\diamond, f : \varphi \Rightarrow \psi, g : \psi \Rightarrow \theta \vdash \lambda x : \varphi.\ g\,(f\, x) \ : \varphi \Rightarrow \theta$

Simply-Typed Lambda Calculus (STLC)

Types: $A, B, C, \ldots ::=$

$G, G', G'' \ldots$	"ground" types
$\mathtt{unit}$	unit type
$A \times B$	product type
$A \to B$	function type

Simply-Typed Lambda Calculus (STLC)

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$A \times B$	product type
$A \to B$	function type

Terms: $s, t, r, \ldots ::=$

c^A	constants (of given type A)
x	variable (countably many)
$()$	unit value
$(s\ ,\ t)$	pair
$\texttt{fst}\ t \quad \texttt{snd}\ t$	projections
$\lambda x : A.\ t$	function abstraction
$s\ t$	function application

STLC

Some examples of terms:

- $\lambda z : (A \to B) \times (A \to C).\, \lambda x : A.\, ((\mathtt{fst}\, z)\, x\, ,\, (\mathtt{snd}\, z)\, x))$

  (has type $((A \to B) \times (A \to C)) \to (A \to (B \times C))$)

- $\lambda z : A \to (B \times C).\, (\lambda x : A.\ \mathtt{fst}(z\, x)\, ,\, \lambda y : A.\ \mathtt{snd}(z\, y))$

  (has type $(A \to (B \times C)) \to ((A \to B) \times (A \to C))$)

- $\lambda z : A \to (B \times C).\, \lambda x : A.\, ((\mathtt{fst}\, z)\, x\, ,\, (\mathtt{snd}\, z)\, x)$

  (has no type)

STLC typing relation, $\Gamma \vdash t : A$

Γ ranges over typing environments

$$\Gamma ::= \diamond \mid \Gamma, x : A$$

(so typing environments are comma-separated snoc-lists of (variable,type)-pairs

— in fact only the lists whose variables are mutually distinct get used)

The typing relation $\Gamma \vdash t : A$ is inductively defined by the following rules, which make use of the following notation

$\boxed{\Gamma\ \mathtt{ok}}$ means: no variable occurs more than once in Γ

$\boxed{\mathtt{dom}\,\Gamma}$ = finite set of variables occurring in Γ

STLC typing relation, $\Gamma \vdash t : A$

Typing rules for variables

$$\frac{\Gamma\ \texttt{ok} \qquad x \notin \texttt{dom}\,\Gamma}{\Gamma, x : A \vdash x : A}\ (\textsc{var})$$

$$\frac{\Gamma \vdash x : A \qquad x' \notin \texttt{dom}\,\Gamma}{\Gamma, x' : A' \vdash x : A}\ (\textsc{var'})$$

Typing rules for constants and unit value

$$\frac{\Gamma\ \texttt{ok}}{\Gamma \vdash c^A : A}\ (\textsc{cons})$$

$$\frac{\Gamma\ \texttt{ok}}{\Gamma \vdash () : \texttt{unit}}\ (\textsc{unit})$$

STLC typing relation, $\Gamma \vdash t : A$

Typing rules for pairs and projections

$$\frac{\Gamma \vdash s : A \qquad \Gamma \vdash t : B}{\Gamma \vdash (s\,,t) : A \times B}\ (\textsc{pair})$$

$$\frac{\Gamma \vdash t : A \times B}{\Gamma \vdash \mathtt{fst}\, t : A}\ (\textsc{fst})$$

$$\frac{\Gamma \vdash t : A \times B}{\Gamma \vdash \mathtt{snd}\, t : B}\ (\textsc{snd})$$

STLC typing relation, $\Gamma \vdash t : A$

Typing rules for function abstraction & application

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x : A.\, t : A \to B}\ (\textsc{fun})$$

$$\frac{\Gamma \vdash s : A \to B \qquad \Gamma \vdash t : A}{\Gamma \vdash s\, t : B}\ (\textsc{app})$$

STLC typing relation, $\Gamma \vdash t : A$

Example typing derivation:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash g : B \to C}\,(\textsc{var})}{\Gamma, x : A \vdash g : B \to C}\,(\textsc{var'}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\diamond, f : A \to B \vdash f : A \to B}\,(\textsc{var})}{\Gamma \vdash f : A \to B}\,(\textsc{var'})}{\Gamma, x : A \vdash f : A \to B}\,(\textsc{var'}) \qquad \dfrac{}{\Gamma, x : A \vdash x : A}\,(\textsc{var})}{\Gamma, x : A \vdash f\, x : B}\,(\textsc{app})}{\Gamma, x : A \vdash g(f\, x) : C}\,(\textsc{app})}{\Gamma \vdash \lambda x : A.\, g(f\, x) : A \to C}\,(\textsc{fun})}{\diamond, f : A \to B \vdash \lambda g : B \to C.\, \lambda x : A.\, g(f\, x) : (B \to C) \to (A \to C)}\,(\textsc{fun})}{\diamond \vdash \lambda f : A \to B.\, \lambda g : B \to C.\, \lambda x : A.\, g(f\, x) : (A \to B) \to (B \to C) \to (A \to C)}\,(\textsc{fun})
$$

where $\Gamma \triangleq \diamond, f : A \to B, g : B \to C$

N.B. the STLC typing rules are "syntax-directed", by the structure of terms t and then in the case of variables x, by the structure of typing environments Γ.

Semantics of STLC types in a ccc

Given a cartesian closed category $\mathbf{C}$,

any function M mapping ground types G to objects $M(G) \in \mathbf{C}$

extends to function $A \mapsto M[\![A]\!] \in \mathbf{C}$ and $\Gamma \mapsto M[\![\Gamma]\!] \in \mathbf{C}$ from STLC types and typing environments to $\mathbf{C}$-objects, by recursion on the structure of A:

$$
\begin{aligned}
M[\![G]\!] &= M(G) \\
M[\![\mathtt{unit}]\!] &= 1 && \text{terminal object in } \mathbf{C} \\
M[\![A \times B]\!] &= M[\![A]\!] \times M[\![B]\!] && \text{product in } \mathbf{C} \\
M[\![A \to B]\!] &= M[\![A]\!] \to M[\![B]\!] && \text{exponential in } \mathbf{C}
\end{aligned}
$$

$$
\begin{aligned}
M[\![\diamond]\!] &= 1 && \text{terminal object in } \mathbf{C} \\
M[\![\Gamma, x : A]\!] &= M[\![\Gamma]\!] \times M[\![A]\!] && \text{product in } \mathbf{C}
\end{aligned}
$$