Lecture 6

L108 assessment—heads up

Graded exercise sheet (Ex.Sh.#4) for 25% credit

- issued Friday 29 October 2021 via Moodle
- your answers are due (via Moodle) by 12:00 on Friday 5 November 2021

Take-home exam, 75% credit, will be available via Moodle from 12:00 on Friday 26 November 2021, with solutions to be submitted by 12:00 on Friday 3 December 2021.

ccc

Recall:

Definition. $\mathbf{C}$ is a cartesian closed category (ccc) if it is a category with a terminal object, binary products and exponentials of any pair of objects.

Non-example of a ccc

The category **Mon** of monoids has a terminal object and binary products, but is not a ccc

because of the following bijections between sets, where 1 denotes a one-element set and the corresponding one-element monoid:

$$\mathbf{Set}(1, \mathtt{List}\,1) \cong \mathbf{Mon}(\mathtt{List}\,1, \mathtt{List}\,1)$$
$$\cong \mathbf{Mon}(1 \times \mathtt{List}\,1, \mathtt{List}\,1)$$

by universal property of the free monoid $\mathtt{List}\,1$ on the one-element set 1

by Ex.Sh. 2, qu. 2 (1 is terminal in **Mon**)

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because of the following bijections between sets, where $\mathtt{1}$ denotes a one-element set and the corresponding one-element monoid:

$$\begin{aligned}\mathbf{Set}(\mathtt{1}, \mathtt{List}\,\mathtt{1}) &\cong \mathbf{Mon}(\mathtt{List}\,\mathtt{1}, \mathtt{List}\,\mathtt{1}) \\ &\cong \mathbf{Mon}(\mathtt{1} \times \mathtt{List}\,\mathtt{1}, \mathtt{List}\,\mathtt{1})\end{aligned}$$

Since $\mathbf{Set}(\mathtt{1}, \mathtt{List}\,\mathtt{1})$ is countably infinite, so is $\mathbf{Mon}(\mathtt{1} \times \mathtt{List}\,\mathtt{1}, \mathtt{List}\,\mathtt{1})$.

Since the one-element monoid is initial (see Lect. 3) in **Mon**, for any $M \in \mathbf{Mon}$ we have that $\mathbf{Mon}(\mathtt{1}, M)$ has just one element and hence

$$\mathbf{Mon}(\mathtt{1} \times \mathtt{List}\,\mathtt{1}, \mathtt{List}\,\mathtt{1}) \ncong \mathbf{Mon}(\mathtt{1}, M)$$

Therefore no M can be the exponential of the objects $\mathtt{List}\,\mathtt{1}$ and $\mathtt{List}\,\mathtt{1}$ in **Mon**.

Cartesian closed pre-order

Recall that each pre-ordered set $(P, \sqsubseteq)$ gives a category $\mathbf{C}_P$. It is a ccc iff P has

- a greatest element $\top$: $\forall p \in P,\ p \sqsubseteq \top$
- binary meets $p \wedge q$:
  $\forall r \in P,\ r \sqsubseteq p \wedge q \;\Leftrightarrow\; r \sqsubseteq p \wedge r \sqsubseteq q$
- Heyting implications $p \rightarrow q$:
  $\forall r \in P,\ r \sqsubseteq p \rightarrow q \;\Leftrightarrow\; r \wedge p \sqsubseteq q$

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- binary meets $p \wedge q$:
  $\forall r \in P,\ r \sqsubseteq p \wedge q \iff r \sqsubseteq p \wedge r \sqsubseteq q$
- Heyting implications $p \to q$:
  $\forall r \in P,\ r \sqsubseteq p \to q \iff r \wedge p \sqsubseteq q$

E.g. any Boolean algebra (with $p \to q = \neg p \vee q$).

E.g. $([0,1], \leq)$ with $\top = 1$, $p \wedge q = \min\{p, q\}$ and $p \to q = \begin{cases} 1 & \text{if } p \leq q \\ q & \text{if } q < p \end{cases}$

Intuitionistic Propositional Logic (IPL)

We present it in "natural deduction" style and only consider the fragment with conjunction and implication, with the following syntax:

Formulas of IPL: $\varphi, \psi, \theta, \ldots ::=$

$p, q, r, \ldots$	propositional identifiers
$\texttt{true}$	truth
$\varphi \mathbin{\&} \psi$	conjunction
$\varphi \Rightarrow \psi$	implication

Sequents of IPL: $\Phi ::= \diamond$ empty

$\qquad\qquad\qquad\qquad \mid \Phi, \phi$ non=empty

(so sequents are finite snoc-lists of formulas)

IPL entailment $\Phi \vdash \varphi$

The intended meaning of $\Phi \vdash \varphi$ is "the conjunction of the formulas in Φ implies the formula φ". The relation $\_ \vdash \_$ is inductively generated by the following rules:

$$\frac{}{\Phi, \varphi \vdash \varphi}\ (\textsc{ax}) \qquad \frac{\Phi \vdash \varphi}{\Phi, \psi \vdash \varphi}\ (\textsc{wk}) \qquad \frac{\Phi \vdash \varphi \qquad \Phi, \varphi \vdash \psi}{\Phi \vdash \psi}\ (\textsc{cut})$$

$$\frac{}{\Phi \vdash \mathtt{true}}\ (\textsc{true}) \qquad \frac{\Phi \vdash \varphi \qquad \Phi \vdash \psi}{\Phi \vdash \varphi \,\&\, \psi}\ (\&\textsc{i}) \qquad \frac{\Phi, \varphi \vdash \psi}{\Phi \vdash \varphi \Rightarrow \psi}\ ({\Rightarrow}\textsc{i})$$

$$\frac{\Phi \vdash \varphi \,\&\, \psi}{\Phi \vdash \varphi}\ (\&\textsc{e}_1) \qquad \frac{\Phi \vdash \varphi \,\&\, \psi}{\Phi \vdash \psi}\ (\&\textsc{e}_2) \qquad \frac{\Phi \vdash \varphi \Rightarrow \psi \qquad \Phi \vdash \varphi}{\Phi \vdash \psi}\ ({\Rightarrow}\textsc{e})$$

For example, if $\Phi = \diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta$, then $\Phi \vdash \varphi \Rightarrow \theta$ is provable in IPL, because:

$$
\dfrac{
  \dfrac{
    \dfrac{}{\Phi \vdash \psi \Rightarrow \theta}\,(\textsc{ax})
  }{\Phi, \varphi \vdash \psi \Rightarrow \theta}\,(\textsc{wk})
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{}{\diamond, \varphi \Rightarrow \psi \vdash \varphi \Rightarrow \psi}\,(\textsc{ax})
      }{\Phi \vdash \varphi \Rightarrow \psi}\,(\textsc{wk})
    }{\Phi, \varphi \vdash \varphi \Rightarrow \psi}\,(\textsc{wk})
    \qquad
    \dfrac{}{\Phi, \varphi \vdash \varphi}\,(\textsc{ax})
  }{\Phi, \varphi \vdash \psi}\,({\Rightarrow}\textsc{e})
}{
  \dfrac{\Phi, \varphi \vdash \theta}{\Phi \vdash \varphi \Rightarrow \theta}\,({\Rightarrow}\textsc{i})
}\,({\Rightarrow}\textsc{e})
$$

Semantics of IPL

in a cartesian closed pre-oder $(P, \sqsubseteq)$

Given a function M assigning a meaning to each propositional identifier p as an element $M(p) \in P$, we can assign meanings to IPL formula φ and sequents Φ as element $M[\![\varphi]\!], M[\![\Phi]\!] \in P$ by recursion on their structure:

$M[\![p]\!] = M(p)$	
$M[\![\mathtt{true}]\!] = \top$	greatest element
$M[\![\varphi \,\&\, \psi]\!] = M[\![\varphi]\!] \wedge M[\![\psi]\!]$	binary meet
$M[\![\varphi \Rightarrow \psi]\!] = M[\![\varphi]\!] \rightarrow M[\![\psi]\!]$	Heyting implication
$M[\![\diamond]\!] = \top$	greatest element
$M[\![\Phi, \varphi]\!] = M[\![\Phi]\!] \wedge M[\![\varphi]\!]$	binary meet

Semantics of IPL

in a cartesian closed pre-oder $(P, \sqsubseteq)$

Soundness Theorem. If $\Phi \vdash \varphi$ is provable from the rules of IPL, then $M[\![\Phi]\!] \sqsubseteq M[\![\varphi]\!]$ holds in any cartesian closed pre-order.

Proof. exercise (show that $\{(\Phi, \varphi) \mid M[\![\Phi]\!] \sqsubseteq M[\![\varphi]\!]\}$ is closed under the rules defining IPL entailment and hence contains $\{(\Phi, \varphi) \mid \Phi \vdash \varphi\}$)

Example

Peirce's Law $\diamond \vdash ((\varphi \Rightarrow \psi) \Rightarrow \varphi) \Rightarrow \varphi$

is not provable in IPL.

(whereas the formula $((\varphi \Rightarrow \psi) \Rightarrow \varphi) \Rightarrow \varphi$ is a classical tautology)

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For if $\diamond \vdash ((\varphi \Rightarrow \psi) \Rightarrow \varphi) \Rightarrow \varphi$ were provable in IPL, then by the Soundness Theorem we would have
$\top = M[\![\diamond]\!] \sqsubseteq M[\![((\varphi \Rightarrow \psi) \Rightarrow \varphi) \Rightarrow \varphi]\!]$.

But in the cartesian closed partial order $([0,1], \leq)$, taking $M(p) = 1/2$ and $M(q) = 0$, we get

$$
\begin{aligned}
M[\![((p \Rightarrow q) \Rightarrow p) \Rightarrow p]\!] &= ((1/2 \to 0) \to 1/2) \to 1/2 \\
&= (0 \to 1/2) \to 1/2 \\
&= 1 \to 1/2 \\
&= 1/2 \\
&\not\geq 1
\end{aligned}
$$

Semantics of IPL

in a cartesian closed pre-oder $(P, \sqsubseteq)$

Completeness Theorem. Given Φ, φ, if for all cartesian closed pre-orders $(P, \sqsubseteq)$ and all interpretations M of the propositional identifiers as elements of P, it is the case that $M[\![\Phi]\!] \sqsubseteq M[\![\varphi]\!]$ holds in P, then $\Phi \vdash \varphi$ is provable in IPL.

Semantics of IPL

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Completeness Theorem. Given Φ, φ, if for all cartesian closed pre-orders $(P, \sqsubseteq)$ and all interpretations M of the propositional identifiers as elements of P, it is the case that $M[\![\Phi]\!] \sqsubseteq M[\![\varphi]\!]$ holds in P, then $\Phi \vdash \varphi$ is provable in IPL.

Proof. Define

$$\begin{aligned} P &\triangleq \{\text{formulas of IPL}\} \\ \varphi \sqsubseteq \psi &\triangleq \diamond, \varphi \vdash \psi \text{ is provable in IPL} \end{aligned}$$

Then one can show that $(P, \sqsubseteq)$ is a cartesian closed pre-ordered set.
For this $(P, \sqsubseteq)$, taking M to be $M(p) = p$, one can show that $M[\![\Phi]\!] \sqsubseteq M[\![\varphi]\!]$ holds in P iff $\Phi \vdash \varphi$ is provable in IPL. □