Lecture 5

Exponentials

Given $X, Y \in Set$, let $Y^X \in Set$ denote the set of all functions from X to Y.

 $Y^X = \mathbf{Set}(X, Y) = \{ f \subseteq X \times Y \mid f \text{ is single-valued and total} \}$

Aim to characterise Y^X category theoretically.

Exponentials

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Function application gives a morphism app : $Y^X \times X \rightarrow Y$ in Set.

$$app(f, x) = f x$$
 $(f \in Y^X, x \in X)$

so as a set of ordered pairs, app is $\{((f, x), y) \in (Y^X \times X) \times Y \mid (x, y) \in f\}$

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Currying operation transforms morphisms $f: Z \times X \rightarrow Y$ in Set to morphisms $\operatorname{cur} f: Z \rightarrow Y^X$

$$\operatorname{cur} f z x = f(z, x) \qquad (f \in Y^X, z \in Z, x \in X)$$

 $cur f z = \{(x, y) \mid ((z, x), y) \in f\}$ $cur f = \{(z, g) \mid g = \{(x, y) \mid ((z, x), y) \in f\}\}$

Haskell Curry

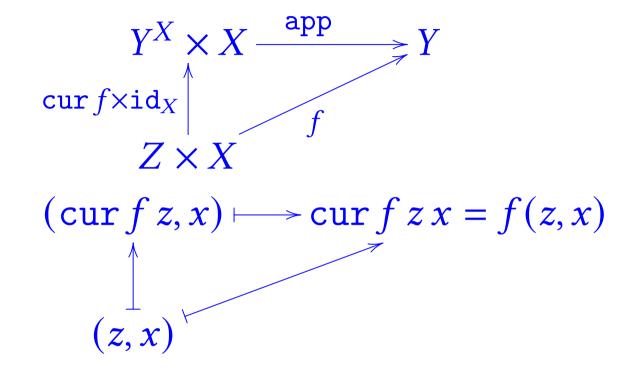
Haskell Brooks Curry

(/ˈhæskəl/; September 12, 1900 – September 1, 1982) was an American mathematician and logician. Curry is best known for his work in combinatory logic; while the initial concept of combinatory logic was based on a single paper by Moses Schönfinkel,^[1] much of the development was done by Curry. Curry is also known for <u>Curry's</u> paradox and the <u>Curry</u>– Howard correspondence. There are three programming languages named after him, Haskell, Brook and Curry, as well as the concept of *curruing*, a

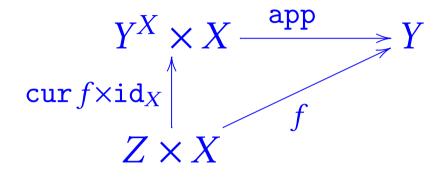
Haskell Brooks Curry

Born	September 12, 1900 <u>Millis, Massachusetts</u>
Died	September 1, 1982 (aged 81) <u>State College, Pennsylvania</u>
Nationality	American
Alma mater	Harvard University
Known for	<u>Combinatory logic</u> <u>Curry–Howard</u> correspondence

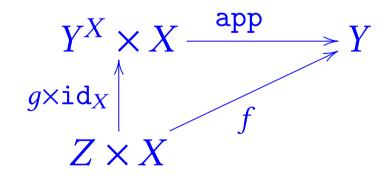
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Furthermore, if any function $g: Z \rightarrow Y^X$ also satisfies



then $g = \operatorname{cur} f$, because of function extensionality...

Function Extensionality

Two functions $f, g \in Y^X$ are equal if (and only if) $\forall x \in X, f x = g x.$

This is true of the set-theoretic notion of function, because then

i.e.
$$\{(x, f x) \mid x \in X\} = \{(x, g x) \mid x \in X\}$$

i.e.
$$\{(x, y) \mid (x, y) \in f\} = \{(x, y) \mid (x, y) \in g\}$$

i.e.
$$f = g$$

(in other words it reduces to the extensionality property of sets: two sets are equal iff they have the same elements).

Suppose a category **C** has binary products, that is, for every pair of **C**-objects *X* and *Y* there is a product diagram $X \xleftarrow{\pi_1} X \times Y \xrightarrow{\pi_2} Y$.

Notation: given $f \in C(X, X')$ and $f' \in C(Y, Y')$, then $f \times f' : X \times Y \to X' \times Y'$ stands for $\langle f \circ \pi_1, f' \circ \pi_2 \rangle$, that is, the unique morphism $g \in C(X \times Y, X' \times Y')$ satisfying $\pi_1 \circ g = f \circ \pi_1$ and $\pi_2 \circ g = f' \circ \pi_2$.

L5

Suppose a category C has binary products. An exponential for C-objects X and Y is specified by object Y^X + morphism app : $Y^X \times X \rightarrow Y$ satisfying the universal property for all $Z \in \mathbb{C}$ and $f \in \mathbb{C}(Z \times X, Y)$, there is a unique $g \in \mathbf{C}(Z, Y^X)$ such that $Y^X \times X \xrightarrow{app} g \times \mathrm{id}_X$ $Z \times X$ commutes in C.

Notation: we write $\operatorname{cur} f$ for the unique g such that $\operatorname{app} \circ (g \times \operatorname{id}_X) = f$.

The universal property of app : $Y^X \times X \rightarrow Y$ says that there is a bijection

 $C(Z, Y^X) \cong C(Z \times X, Y)$ $g \mapsto \operatorname{app} \circ (g \times \operatorname{id}_X)$ $\operatorname{cur} f \leftarrow f$ $\operatorname{app} \circ (\operatorname{cur} f \times \operatorname{id}_X) = f$ $g = \operatorname{cur}(\operatorname{app} \circ (g \times \operatorname{id}_X))$

The universal property of app : $Y^X \times X \rightarrow Y$ says that there is a bijection...

It also says that (Y^X, app) is a terminal object in the following category:

- objects: (Z, f) where $f \in \mathbf{C}(Z \times X, Y)$
- morphisms $g: (Z, f) \to (Z', f')$ are $g \in C(Z, Z')$ such that $f' \circ (g \times id_X) = f$
- composition and identities as in C.

So when they exist, exponential objects are unique up to (unique) isomorphism.

Cartesian closed category

Definition. C is a cartesian closed category (ccc) if it is a category with a terminal object, binary products and exponentials of any pair of objects.

This is a key concept for the semantics of lambda calculus and for the foundations of functional programming languages.

Notation: an exponential object Y^X is often written as $X \to Y$

Cartesian closed category

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Examples:

- ▶ Set is a ccc as we have seen.
- ▶ **Preord** is a ccc: we already saw that it has a terminal object and binary products; the exponential of (P_1, \sqsubseteq_1) and (P_2, \sqsubseteq_2) is $(P_1 \rightarrow P_2, \sqsubseteq)$ where

 $P_1 \to P_2 = \mathbf{Preord}((P_1, \sqsubseteq_1), (P_2, \bigsqcup_2))$ $f \sqsubseteq g \iff \forall x \in P_1, \ f \ x \sqsubseteq_2 g \ x$

(check that this is a pre-order and does give an exponential in **Preord**)