Category Theory

Lecture 15

Presheaf categories

Let $\mathbf{C}$ be a small category. The functor category $\mathbf{Set}^{\mathbf{C}^{op}}$ is called the category of presheaves on $\mathbf{C}$.

- objects are contravariant functors from $\mathbf{C}$ to $\mathbf{Set}$
- morphisms are natural transformations

Much used in the semantics of various dependently-typed languages and logics.

Given category $\mathbb{C}$
with terminal object 1

global elements of some $X \in \mathrm{obj}\,\mathbb{C}$
are by definition the morphisms
$1 \longrightarrow X$ in $\mathbb{C}$

e.g. in $\mathbf{Set}$ ...

but in $\mathbf{Mon}$ ...

Given category $\mathbb{C}$
with terminal object 1

> global elements of some $X \in \mathrm{obj}\,\mathbb{C}$
> are by definition the morphisms
> $1 \longrightarrow X$ in $\mathbb{C}$

We say $\mathbb{C}$ is well-pointed if for all $X \overset{f}{\underset{g}{\rightrightarrows}} Y$
in $\mathbb{C}$ we have:

$$\left(\forall\, 1 \xrightarrow{x} X,\ f \circ x = g \circ x\right) \Rightarrow f = g$$

($\mathbf{Set}$ is, $\mathbf{Mon}$ isn't)

Idea

replace global elements $1 \xrightarrow{x} X$

by $Y \xrightarrow{x} X$ (any $Y \in \mathrm{Obj}\,\mathbb{C}$)

"$x \in_Y X$" "x is a **generalised element** of X at stage Y"

Have to take into account change of stage:

$$x \in_Y X \ \& \ Z \xrightarrow{f} Y \ \rightsquigarrow \ x \circ f \in_Z X$$

Presheaf categories

Let **C** be a small category. The functor category $\mathbf{Set}^{\mathbf{C}^{op}}$ is called the category of presheaves on **C**.

- objects are contravariant functors from **C** to **Set**
- morphisms are natural transformations

Much used in the semantics of various dependently-typed languages and logics.

Yoneda functor

$$よ : \mathbf{C} \rightarrow \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$$

(where $\mathbf{C}$ is a small category)

is the Curried version of the hom functor

$$\mathbf{C} \times \mathbf{C}^{\mathrm{op}} \cong \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \xrightarrow{\mathtt{Hom}_{\mathbf{C}}} \mathbf{Set}$$

Yoneda functor

$$よ : \mathbf{C} \longrightarrow \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$$

(where $\mathbf{C}$ is a small category)

is the Curried version of the hom functor

$$\mathbf{C} \times \mathbf{C}^{\mathrm{op}} \cong \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \xrightarrow{\mathtt{Hom}_{\mathbf{C}}} \mathbf{Set}$$

- For each $\mathbf{C}$-object X, the object $よX \in \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ is the functor $\mathbf{C}(\_,X) : \mathbf{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ given by

$$\begin{array}{cccc} Z & \mapsto & \mathbf{C}(Z,X) & g \circ f \\ \downarrow f & \mapsto & \uparrow & \uparrow \\ Y & \mapsto & \mathbf{C}(Y,X) & g \end{array}$$

Yoneda functor

$$\text{よ} : \mathbf{C} \longrightarrow \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$$

(where $\mathbf{C}$ is a small category)

is the Curried version of the hom functor

$$\mathbf{C} \times \mathbf{C}^{\mathrm{op}} \cong \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \xrightarrow{\mathrm{Hom}_{\mathbf{C}}} \mathbf{Set}$$

- For each $\mathbf{C}$-object X, the object $\text{よ}X \in \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ is the functor $\mathbf{C}(\_, X) : \mathbf{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ given by

$$\begin{array}{ccccc} Z & \mapsto & \mathbf{C}(Z, X) & & g \circ f \\ \downarrow f & \mapsto & \uparrow & & \uparrow f^* \\ Y & \mapsto & \mathbf{C}(Y, X) & & g \end{array}$$

this function is often written as f^*

Yoneda functor

$$\text{よ} : \mathbf{C} \to \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$$

(where $\mathbf{C}$ is a small category)

is the Curried version of the hom functor

$$\mathbf{C} \times \mathbf{C}^{\mathrm{op}} \cong \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \xrightarrow{\mathtt{Hom}_{\mathbf{C}}} \mathbf{Set}$$

- For each $\mathbf{C}$-morphism $Y \xrightarrow{f} X$, the morphism $\text{よ}Y \xrightarrow{\text{よ}f} \text{よ}X$ in $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ is the natural transformation whose component at any given $Z \in \mathbf{C}^{\mathrm{op}}$ is the function

$$\begin{array}{ccc} \text{よ}Y(Z) & \xrightarrow{(\text{よ}f)_Z} & \text{よ}X(Z) \\ \| & & \| \\ \mathbf{C}(Z, Y) & & \mathbf{C}(Z, X) \\ g & \longmapsto & f \circ g \end{array}$$

Yoneda functor

$$\text{よ} : \mathbf{C} \to \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$$

(where $\mathbf{C}$ is a small category)

is the Curried version of the hom functor

$$\mathbf{C} \times \mathbf{C}^{\mathrm{op}} \cong \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \xrightarrow{\mathrm{Hom}_{\mathbf{C}}} \mathbf{Set}$$

- For each $\mathbf{C}$-morphism $Y \xrightarrow{f} X$, the morphism $\text{よ}Y \xrightarrow{\text{よ}f} \text{よ}X$ in $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ is the natural transformation whose component at any given $Z \in \mathbf{C}^{\mathrm{op}}$ is the function

$$\begin{array}{ccc} \text{よ}Y(Z) & \xrightarrow{(\text{よ}f)_Z} & \text{よ}X(Z) \\ \| & & \| \\ \mathbf{C}(Z,Y) & & \mathbf{C}(Z,X) \\ g & \xmapsto{f_*} & f \circ g \end{array}$$

this function is often written as f_*

The Yoneda Lemma

For each small category $\mathbf{C}$, each object $X \in \mathbf{C}$ and each presheaf $F \in \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$, there is a bijection of sets

$$\eta_{X,F} : \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\text{よ}X, F) \cong F(X)$$

which is natural in both X and F.

The Yoneda Lemma

For each small category $\mathbf{C}$, each object $X \in \mathbf{C}$ and each presheaf $F \in \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$, there is a bijection of sets

$$\eta_{X,F} : \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\text{よ}X, F) \cong F(X)$$

which is natural in both X and F.

the value of $F : \mathbf{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ at X

the set of natural transformations from the functor $\text{よ}X : \mathbf{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ to the functor $F : \mathbf{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$

The Yoneda Lemma

For each small category $\mathbf{C}$, each object $X \in \mathbf{C}$ and each presheaf $F \in \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$, there is a bijection of sets

$$\eta_{X,F} : \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(よX, F) \cong F(X)$$

which is natural in both X and F.

Definition of the function $\eta_{X,F} : \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(よX, F) \to F(X)$:

for each $\theta : よX \to F$ in $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ we have the function $\mathbf{C}(X, X) = よX(X) \xrightarrow{\theta_X} F(X)$ and define

$$\boxed{\eta_{X,F}(\theta) \triangleq \theta_X(\mathtt{id}_X)}$$

The Yoneda Lemma

For each small category $\mathbf{C}$, each object $X \in \mathbf{C}$ and each presheaf $F \in \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$, there is a bijection of sets

$$\eta_{X,F} : \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\text{よ}X, F) \cong F(X)$$

which is natural in both X and F.

Definition of the function $\eta^{-1}_{X,F} : F(X) \to \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\text{よ}X, F)$:

for each $x \in F(X)$, $Y \in \mathbf{C}$ and $f \in \text{よ}X(Y) = \mathbf{C}(Y, X)$,

we get a $F(X) \xrightarrow{F(f)} F(Y)$ in $\mathbf{Set}$ and hence $F(f)(x) \in F(Y)$;

The Yoneda Lemma

For each small category $\mathbf{C}$, each object $X \in \mathbf{C}$ and each presheaf $F \in \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$, there is a bijection of sets

$$\eta_{X,F} : \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\text{よ}X, F) \cong F(X)$$

which is natural in both X and F.

Definition of the function $\eta_{X,F}^{-1} : F(X) \to \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\text{よ}X, F)$:

for each $x \in F(X)$, $Y \in \mathbf{C}$ and $f \in \text{よ}X(Y) = \mathbf{C}(Y, X)$,

we get a $F(X) \xrightarrow{F(f)} F(Y)$ in $\mathbf{Set}$ and hence $F(f)(x) \in F(Y)$;

Define $\left(\eta_{X,F}^{-1}(x)\right)_Y : \text{よ}X(Y) \to F(Y)$ by

$$\left(\eta_{X,F}^{-1}(x)\right)_Y (f) \triangleq F(f)(x)$$

check this gives a natural transformation $\eta_{X,F}^{-1}(x) : \text{よ}X \to F$

Proof of $\eta_{X,F} \circ \eta_{X,F}^{-1} = \mathtt{id}_{F(X)}$

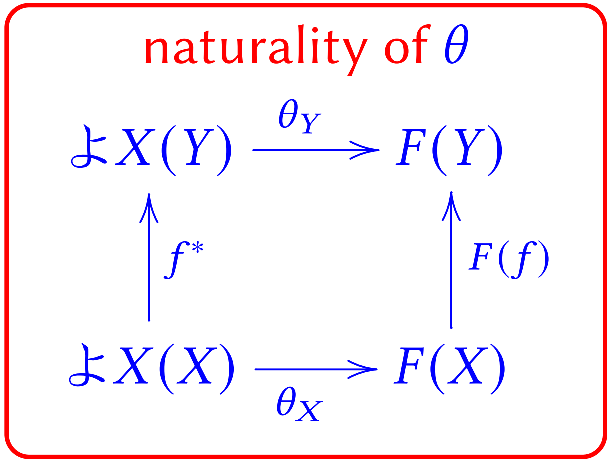
For any $x \in F(X)$ we have

$$
\begin{aligned}
\eta_{X,F}\left(\eta_{X,F}^{-1}(x)\right) &\triangleq \left(\eta_{X,F}^{-1}(x)\right)_X(\mathtt{id}_X) && \text{by definition of } \eta_{X,F}\\
&\triangleq F(\mathtt{id}_X)(x) && \text{by definition of } \eta_{X,F}^{-1}\\
&= \mathtt{id}_{F(X)}(x) && \text{since } F \text{ is a functor}\\
&= x
\end{aligned}
$$

Proof of $\eta^{-1}_{X,F} \circ \eta_{X,F} = \mathtt{id}_{\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(よX,F)}$

For any $よX \xrightarrow{\theta} F$ in $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ and $Y \xrightarrow{f} X$ in $\mathbf{C}$, we have

$$
\begin{aligned}
\left(\eta^{-1}_{X,F}\left(\eta_{X,F}(\theta)\right)\right)_Y f &\triangleq \left(\eta^{-1}_{X,F}\left(\theta_X(\mathtt{id}_X)\right)\right)_Y f && \text{by definition of } \eta_{X,F} \\
&\triangleq F(f)(\theta_X(\mathtt{id}_X)) && \text{by definition of } \eta^{-1}_{X,F} \\
&= \theta_Y(f^*(id_X)) && \text{by naturality of } \theta \\
&\triangleq \theta_Y(\mathtt{id}_X \circ f) && \text{by definition of } f^* \\
&= \theta_Y(f)
\end{aligned}
$$



Proof of $\eta^{-1}_{X,F} \circ \eta_{X,F} = \mathtt{id}_{\mathbf{Set}^{\mathbf{C}^{op}}(よX,F)}$

For any $よX \xrightarrow{\theta} F$ in $\mathbf{Set}^{\mathbf{C}^{op}}$ and $Y \xrightarrow{f} X$ in $\mathbf{C}$, we have

$$\begin{aligned}
\left(\eta^{-1}_{X,F}\left(\eta_{X,F}(\theta)\right)\right)_Y f &\triangleq \left(\eta^{-1}_{X,F}\left(\theta_X(\mathtt{id}_X)\right)\right)_Y f && \text{by definition of } \eta_{X,F} \\
&\triangleq F(f)(\theta_X(\mathtt{id}_X)) && \text{by definition of } \eta^{-1}_{X,F} \\
&= \theta_Y(f^*(id_X)) && \text{by naturality of } \theta \\
&\triangleq \theta_Y(\mathtt{id}_X \circ f) && \text{by definition of } f^* \\
&= \theta_Y(f)
\end{aligned}$$

so $\forall \theta, Y, \ \left(\eta^{-1}_{X,F}\left(\eta_{X,F}(\theta)\right)\right)_Y = \theta_Y$

so $\forall \theta, \ \eta^{-1}_{X,F}\left(\eta_{X,F}(\theta)\right) = \theta$

so $\eta^{-1}_{X,F} \circ \eta_{X,F} = \mathtt{id}$.

The Yoneda Lemma

For each small category $\mathbf{C}$, each object $X \in \mathbf{C}$ and each presheaf $F \in \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$, there is a bijection of sets

$$\eta_{X,F} : \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\text{よ}X, F) \cong F(X)$$

which is natural in both X and F.

Proof that $\eta_{X,F}$ is natural in F:

Given $F \xrightarrow{\varphi} G$ in $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$, have to show that

$$\begin{array}{ccc}
\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\text{よ}X, F) & \xrightarrow{\eta_{X,F}} & F(X) \\
\Big\downarrow{\scriptstyle \varphi_*} & & \Big\downarrow{\scriptstyle \varphi_X} \\
\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\text{よ}X, G) & \xrightarrow[\eta_{X,G}]{} & G(X)
\end{array}$$

commutes in **Set**. For all $\text{よ}X \xrightarrow{\theta} F$ we have

$$\begin{aligned}
\varphi_X\left(\eta_{X,F}(\theta)\right) &\triangleq \varphi_X\left(\theta_X(\mathtt{id}_X)\right) \\
&\triangleq (\varphi \circ \theta)_X(\mathtt{id}_X) \\
&\triangleq \eta_{X,G}(\varphi \circ \theta) \\
&\triangleq \eta_{X,G}(\varphi_*(\theta))
\end{aligned}$$

Proof that $\eta_{X,F}$ is natural in X:

Given $Y \xrightarrow{f} X$ in $\mathbf{C}$, have to show that

$$\begin{array}{ccc}
\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\text{よ}X, F) & \xrightarrow{\quad \eta_{X,F} \quad} & F(X) \\
{\scriptstyle (\text{よ}f)^*} \big\downarrow & & \big\downarrow {\scriptstyle F(f)} \\
\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\text{よ}Y, F) & \xrightarrow[\quad \eta_{Y,F} \quad]{} & F(Y)
\end{array}$$

commutes in $\mathbf{Set}$. For all $\text{よ}X \xrightarrow{\theta} F$ we have

$$\begin{aligned}
F(f)((\eta_{X,F}(\theta)) &\triangleq F(f)(\theta_X(\mathtt{id}_X)) \\
&= \theta_Y(f^*(\mathtt{id}_X)) && \text{by naturality of } \theta \\
&= \theta_Y(f) \\
&= \theta_Y(f_*(\mathtt{id}_Y)) \\
&\triangleq (\theta \circ \text{よ}f)_Y(\mathtt{id}_Y) \\
&\triangleq \eta_{Y,F}(\theta \circ \text{よ}f) \\
&\triangleq \eta_{Y,F}((\text{よ}f)^*(\theta))
\end{aligned}$$

Corollary of the Yoneda Lemma:

the functor $よ : \mathbf{C} \to \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ is full and faithful.

In general, a functor $F : \mathbf{C} \to \mathbf{D}$ is

- faithful if for all $X, Y \in \mathbf{C}$ the function

$$\begin{array}{ccc} \mathbf{C}(X,Y) & \to & \mathbf{D}(F(X),F(Y)) \\ f & \mapsto & F(f) \end{array}$$

  is injective:

$$\forall f, f' \in \mathbf{C}(X,Y),\ F(f) = F(f') \Rightarrow f = f'$$

- full if the above functions are all surjective:

$$\forall g \in \mathbf{D}(F(X),F(Y)), \exists f \in \mathbf{C}(X,Y),\ F(f) = g$$

Corollary of the Yoneda Lemma:

the functor $よ : \mathbf{C} \longrightarrow \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ is full and faithful.

Proof. From the proof of the Yoneda Lemma, for each $F \in \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ we have a bijection

$$F(X) \xrightarrow{(\eta_{X,F})^{-1}} \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(よX, F)$$

By definition of $(\eta_{X,F})^{-1}$, when $F = よY$ the above function is equal to

$$\begin{array}{rcl} よY(X) = \mathbf{C}(X,Y) & \longrightarrow & \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(よX, よY) \\ f & \mapsto & f_* = よf \end{array}$$

So, being a bijection, $f \mapsto よf$ is both injective and surjective; so $よ$ is both faithful and full. □

Recall (for a small category $\mathbf{C}$):

Yoneda functor $よ : \mathbf{C} \to \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$

Yoneda Lemma: there is a bijection $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(よ X, F) \cong F(X)$ which is natural both in $F \in \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ and $X \in \mathbf{C}$.

An application of the Yoneda Lemma:

Theorem. For each small category $\mathbf{C}$, the category $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ of presheaves is cartesian closed.

Theorem. For each small category $\mathbf{C}$, the category $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ of presheaves is cartesian closed.

Theorem. For each small category $\mathbf{C}$, the category $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ of presheaves is cartesian closed.

Proof sketch.

Terminal object in $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ is the functor $1 : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ given by

$$\begin{cases} 1(X) \triangleq \{0\} & \text{terminal object in } \mathbf{Set} \\ 1(f) \triangleq \mathtt{id}_{\{0\}} \end{cases}$$

Theorem. For each small category $\mathbf{C}$, the category $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ of presheaves is cartesian closed.

Proof sketch.

Product of $F, G \in \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ is the functor $F \times G : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ given by

$$\begin{cases} (F \times G)(X) \triangleq F(X) \times G(X) & \text{cartesian product of sets} \\ (F \times G)(f) \triangleq F(f) \times G(f) \end{cases}$$

with projection morphisms $F \xleftarrow{\pi_1} F \times G \xrightarrow{\pi_2} G$ given by the natural transformations whose components at $X \in \mathbf{C}$ are the projection functions $F(X) \xleftarrow{\pi_1} F(X) \times G(X) \xrightarrow{\pi_2} G(X)$.

Theorem. For each small category $\mathbf{C}$, the category $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ of presheaves is cartesian closed.

Proof sketch.

We can work out what the value of the exponential $G^F \in \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ at $X \in \mathbf{C}$ has to be using the Yoneda Lemma:

$$G^F(X) \cong \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\text{よ}X, G^F) \cong \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\text{よ}X \times F, G)$$

Yoneda Lemma

universal property of the exponential

Theorem. For each small category $\mathbf{C}$, the category $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ of presheaves is cartesian closed.

Proof sketch.

We can work out what the value of the exponential $G^F \in \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ at $X \in \mathbf{C}$ has to be using the Yoneda Lemma:

$$G^F(X) \cong \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\text{よ}X, G^F) \cong \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\text{よ}X \times F, G)$$

We take the set $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\text{よ}X \times F, G)$ to be the definition of the value of G^F at X...

Exponential objects in $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$:

$$\boxed{G^F(X) \triangleq \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\text{よ}X \times F, G)}$$

Given $Y \xrightarrow{f} X$ in $\mathbf{C}$, we have $\text{よ}Y \xrightarrow{\text{よ}f} \text{よ}X$ in $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ and hence

$$\begin{array}{rcl} G^F(Y) \triangleq \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\text{よ}Y \times F, G) & \to & \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\text{よ}X \times F, G) \triangleq G^F(X) \\ \theta & \mapsto & \theta \circ (\text{よ}f \times \mathtt{id}_F) \end{array}$$

We define

$$\boxed{G^F(f) \triangleq (\text{よ}f \times \mathtt{id}_F)^*}$$

Have to check that these definitions make G^F ino a functor $\mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$.

Application morphisms in $\mathbf{Set}^{\mathbf{C}^{op}}$:

Given $F, G \in \mathbf{Set}^{\mathbf{C}^{op}}$, the application morphism

$$\mathtt{app} : G^F \times F \to G$$

is the natural transformation whose component at $X \in \mathbf{C}$ is the function

$$(G^F \times F)(X) \triangleq G^F(X) \times F(X) \triangleq \mathbf{Set}^{\mathbf{C}^{op}}(\text{よ}X \times F, G) \times F(X) \xrightarrow{\mathtt{app}_X} G(X)$$

defined by

$$\boxed{\mathtt{app}_X(\theta, x) \triangleq \theta_X(\mathtt{id}_X, x)}$$

Have to check that this is natural in X.

Currying operation in $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$:

$$\left(H \times F \xrightarrow{\theta} G\right) \mapsto \left(H \xrightarrow{\mathtt{cur}\,\theta} G^F\right)$$

Given $H \times F \xrightarrow{\theta} G$ in $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$, the component of $\mathtt{cur}\,\theta$ at $X \in \mathbf{C}$

$$H(X) \xrightarrow{(\mathtt{cur}\,\theta)_X} G^F(X) \triangleq \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\text{よ}X \times F, G)$$

is the function mapping each $z \in H(X)$ to the natural transformation $\text{よ}X \times F \to G$ whose component at $Y \in \mathbf{C}$ is the function

$$(\text{よ}X \times F)(Y) \triangleq \mathbf{C}(Y, X) \times F(Y) \to G(Y)$$

defined by

$$\boxed{((\mathtt{cur}\,\theta)_X(z))_Y(f, y) \triangleq \theta_Y(H(f)(z), y)}$$

Currying operation in $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$:

$$\left(H \times F \xrightarrow{\theta} G\right) \mapsto \left(H \xrightarrow{\mathtt{cur}\,\theta} G^F\right)$$

$$\boxed{((\mathtt{cur}\,\theta)_X(z))_Y(f, y) \triangleq \theta_Y(H(f)(z), y)}$$

Have to check that this is natural in Y,

then that $(\mathtt{cur}\,\theta)_X$ is natural in X,

then that $\mathtt{cur}\,\theta$ is the unique morphism $H \xrightarrow{\varphi} G^F$ in $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ satisfying $\mathtt{app} \circ (\varphi \times \mathtt{id}_F) = \theta$.

Theorem. For each small category $\mathbf{C}$, the category $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ of presheaves is cartesian closed.

So we can interpret simply typed lambda calculus in any presheaf category.

More than that, presheaf categories (usefully) model dependently-typed languages.