Lecture 9
STLC equations

take the form $\Gamma \vdash s = t : A$ where $\Gamma \vdash s : A$ and $\Gamma \vdash t : A$ are provable.

Such an equation is satisfied by the semantics in a ccc if $M[\Gamma \vdash s : A]$ and $M[\Gamma \vdash t : A]$ are equal $C$-morphisms $M[\Gamma] \rightarrow M[A]$.

**Qu:** which equations are always satisfied in any ccc?

**Ans:** $\beta\eta$-equivalence…
STLC $\beta\eta$-Equality

The relation $\Gamma \vdash s =_\beta^\eta t : A$ (where $\Gamma$ ranges over typing environments, $s$ and $t$ over terms and $A$ over types) is inductively defined by the following rules:
STLC $\beta\eta$-Equality

The relation $\Gamma \vdash s =_{\beta\eta} t : A$ (where $\Gamma$ ranges over typing environments, $s$ and $t$ over terms and $A$ over types) is inductively defined by the following rules:

$\beta$-conversions

- $\Gamma, x : A \vdash t : B \quad \Gamma \vdash s : A$
  \[ \Rightarrow \Gamma \vdash (\lambda x : A. t)s =_{\beta\eta} t[s/x] : B \]

- $\Gamma \vdash s : A \quad \Gamma \vdash t : B$
  \[ \Rightarrow \Gamma \vdash \text{fst}(s, t) =_{\beta\eta} s : A \]

- $\Gamma \vdash s : A \quad \Gamma \vdash t : B$
  \[ \Rightarrow \Gamma \vdash \text{snd}(s, t) =_{\beta\eta} t : B \]
STLC $\beta\eta$-Equality

The relation $\Gamma \vdash s =_{\beta\eta} t : A$ (where $\Gamma$ ranges over typing environments, $s$ and $t$ over terms and $A$ over types) is inductively defined by the following rules:

- $\beta$-conversions
- $\eta$-conversions

\[
\begin{align*}
\Gamma \vdash t : A \rightarrow B & \quad x \text{ does not occur in } t \\
\hline
\Gamma \vdash t =_{\beta\eta} (\lambda x : A. \, t \, x) : A \rightarrow B
\end{align*}
\]

\[
\begin{align*}
\Gamma \vdash t : A \times B & \\
\hline
\Gamma \vdash t =_{\beta\eta} (\text{fst } t, \text{snd } t) : A \times B
\end{align*}
\]

\[
\begin{align*}
\Gamma \vdash t : \text{unit} & \\
\hline
\Gamma \vdash t =_{\beta\eta} () : \text{unit}
\end{align*}
\]
STLC $\beta\eta$-Equality

The relation $\Gamma \vdash s =_{\beta\eta} t : A$ (where $\Gamma$ ranges over typing environments, $s$ and $t$ over terms and $A$ over types) is inductively defined by the following rules:

- $\beta$-conversions
- $\eta$-conversions
- congruence rules

\[
\begin{align*}
\Gamma, x : A \vdash t =_{\beta\eta} t' : B \\
\Gamma \vdash \lambda x : A. t =_{\beta\eta} \lambda x : A. t' : A \to B \\
\Gamma \vdash s =_{\beta\eta} s' : A \to B \quad \Gamma \vdash t =_{\beta\eta} t' : A \\
\Gamma \vdash s \, t =_{\beta\eta} s' \, t' : B
\end{align*}
\]

etc
STLC $\beta\eta$-Equality

The relation $\Gamma \vdash s =_{\beta\eta} t : A$ (where $\Gamma$ ranges over typing environments, $s$ and $t$ over terms and $A$ over types) is inductively defined by the following rules:

- $\beta$-conversions
- $\eta$-conversions
- congruence rules
- $=_{\beta\eta}$ is reflexive, symmetric and transitive

\[
\begin{align*}
\Gamma \vdash t : A & \quad \Rightarrow \quad \Gamma \vdash t =_{\beta\eta} t : A \\
\Gamma \vdash s =_{\beta\eta} t : A \\
\Gamma \vdash r =_{\beta\eta} s : A & \quad \Rightarrow \quad \Gamma \vdash r =_{\beta\eta} t : A
\end{align*}
\]
**STLC $\beta\eta$-Equality**

**Soundness Theorem** for semantics of STLC in a ccc. If $\Gamma \vdash s =_{\beta\eta} t : A$ is provable, then in any ccc

$$M[\Gamma \vdash s : A] = M[\Gamma \vdash t : A]$$

are equal $C$-morphisms $M[\Gamma] \to M[A]$.

**Proof** is by induction on the structure of the proof of $\Gamma \vdash s =_{\beta\eta} t : A$.

Here we just check the case of $\beta$-conversion for functions.

So suppose we have $\Gamma, x : A \vdash t : B$ and $\Gamma \vdash s : A$. We have to see that

$$M[\Gamma \vdash (\lambda x : A. t)s : B] = M[\Gamma \vdash t[s/x] : B]$$
Suppose

\[ M[\Gamma] = X \]
\[ M[A] = Y \]
\[ M[B] = Z \]

\[ M[\Gamma, x : A \vdash t : B] = f : X \times Y \to Z \]
\[ M[\Gamma \vdash s : A] = g : X \to Z \]

Then

\[ M[\Gamma \vdash \lambda x : A. t : A \to B] = \mathsf{cur} \ f : X \to Z^Y \]

and hence

\[ M[\Gamma \vdash (\lambda x : A. t)s : B] \]
\[ = \mathsf{app} \circ \langle \mathsf{cur} \ f, g \rangle \]
\[ = \mathsf{app} \circ (\mathsf{cur} \ f \times \mathsf{id}_Y) \circ \langle \mathsf{id}_X, g \rangle \quad \text{since} \quad (a \times b) \circ \langle c, d \rangle = \langle a \circ c, b \circ d \rangle \]
\[ = f \circ \langle \mathsf{id}_X, g \rangle \quad \text{by definition of} \ \mathsf{cur} \ f \]
\[ = M[\Gamma \vdash t[s/x] : B] \quad \text{by the Substitution Theorem} \]

as required.
The internal language of a ccc, $\mathbf{C}$

- one ground type for each $\mathbf{C}$-object $X$
- for each $X \in \mathbf{C}$, one constant $f^X$ for each $\mathbf{C}$-morphism $f : 1 \to X$ (“global element” of the object $X$)

The types and terms of STLC over this language usefully describe constructions on the objects and morphisms of $\mathbf{C}$ using its cartesian closed structure, but in an “element-theoretic” way.

For example...
Example

In any ccc $C$, for any $X, Y, Z \in C$ there is an isomorphism

$$Z^{(X \times Y)} \cong (Z^Y)^X$$
Example

In any ccc $C$, for any $X, Y, Z \in C$ there is an isomorphism

$$Z^{(X \times Y)} \cong (Z^Y)^X$$

which in the internal language of $C$ is described by the terms

$$\diamond \vdash s : ((X \times Y) \to Z) \to (X \to (Y \to Z))$$
$$\diamond \vdash t : (X \to (Y \to Z)) \to ((X \times Y) \to Z)$$

where

$$\begin{align*}
  s \triangleq & \lambda f : (X \times Y) \to Z. \lambda x : X. \lambda y : Y. f(x, y) \\
  t \triangleq & \lambda g : X \to (Y \to Z). \lambda z : X \times Y. g(fst z)(snd z)
\end{align*}$$

and

which satisfy

$$\begin{align*}
  \diamond, f : (X \times Y) \to Z \vdash t(sf) =_{\beta\eta} f \\
  \diamond, g : X \to (Y \to Z) \vdash sg =_{\beta\eta} g
\end{align*}$$
The Soundness Theorem has a converse—completeness.

In fact for a given set of ground types and typed constants there is a single ccc \( F \) (the free ccc for that language) with an interpretation function \( M \) so that \( \Gamma \vdash s =_{\beta\eta} t : A \) is provable iff \( M[\Gamma \vdash s : A] = M[\Gamma \vdash t : A] \) in \( F \).
Free cartesian closed categories

The Soundness Theorem has a converse—completeness.

In fact for a given set of ground types and typed constants there is a single ccc (the free ccc for that language) with an interpretation function $M$ so that $\Gamma \vdash s =_{\beta\eta} t : A$ is provable iff $M[\Gamma \vdash s : A] = M[\Gamma \vdash t : A]$ in $F$.

- $F$-objects are the STLC types over the given set of ground types
- $F$-morphisms $A \to B$ are equivalence classes of STLC terms $t$ satisfying $\diamond \vdash t : A \to B$ (so $t$ is a closed term—it has no free variables) with respect to the equivalence relation equating $s$ and $t$ if $\diamond \vdash s =_{\beta\eta} t : A \to B$ is provable.
- Identity morphism on $A$ is the equivalence class of $\diamond \vdash \lambda x : A. x : A \to A$.
- Composition of a morphism $A \to B$ represented by $\diamond \vdash s : A \to B$ and a morphism $B \to C$ represented by $\diamond \vdash t : B \to C$ is represented by $\diamond \vdash \lambda x : A. t(s x) : A \to C$. 
Curry-Howard correspondence

<table>
<thead>
<tr>
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E.g. IPL *versus* STLC.
Curry-Howard for IPL vs STLC

Proof of $\Diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta$ in IPL

\[
\begin{array}{c}
\frac{\Phi \vdash \psi \Rightarrow \theta}{\Phi \vdash \varphi \Rightarrow \psi} \text{(WK)} \\
\frac{\Phi \vdash \varphi \Rightarrow \psi}{\Phi \vdash \theta} \text{(AX)} \\
\frac{\Phi \vdash \theta}{\Diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta} \text{(⇒I)} \\
\end{array}
\]

where $\Phi = \Diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta, \varphi$
Curry-Howard for IPL vs STLC

and a corresponding STLC term

\[
\frac{\Phi \vdash x : \varphi}{\frac{\Phi \vdash y : \varphi \Rightarrow \psi}{\frac{\Phi \vdash z : \psi \Rightarrow \theta}{\frac{\Phi \vdash y \cdot x : \psi}{\frac{\Phi \vdash z(y \cdot x) : \theta}{\frac{\diamond, y : \varphi \Rightarrow \psi, z : \psi \Rightarrow \theta \vdash \lambda x : \varphi. z(y \cdot x) : \varphi \Rightarrow \theta}}}}}}\]

where \(\Phi = \diamond, y : \varphi \Rightarrow \psi, z : \psi \Rightarrow \theta, x : \varphi\)
Curry-Howard-Lawvere/Lambek correspondence

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E.g. IPL *versus* STLC *versus* CCCs
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E.g. IPL versus STLC versus CCCs

These correspondences can be made into category-theoretic equivalences—we first need to define the notions of **functor** and **natural transformation** in order to define the notion of **equivalence of categories**.