Lecture 5

Exponentials

Given $X, Y \in \mathbf{Set}$, let $Y^X \in \mathbf{Set}$ denote the set of all functions from X to Y.

$$Y^X = \mathbf{Set}(X, Y) = \{f \subseteq X \times Y \mid f \text{ is single-valued and total}\}$$

Aim to characterise Y^X category theoretically.

Exponentials

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Function application gives a morphism $\mathrm{app} : Y^X \times X \to Y$ in $\mathbf{Set}$.

$$\mathrm{app}(f, x) = f\,x \qquad (f \in Y^X, x \in X)$$

so as a set of ordered pairs, app is
$\{((f, x), y) \in (Y^X \times X) \times Y \mid (x, y) \in f\}$

Exponentials

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Function application gives a morphism $\mathrm{app} : Y^X \times X \to Y$ in $\mathbf{Set}$.

Currying operation transforms morphisms $f : Z \times X \to Y$ in $\mathbf{Set}$ to morphisms $\mathtt{cur}\, f : Z \to Y^X$

$$\mathtt{cur}\, f\, z\, x = f(z, x) \qquad (f \in Y^X, z \in Z, x \in X)$$

$$\boxed{\begin{array}{c} \mathtt{cur}\, f\, z = \{(x, y) \mid ((z, x), y) \in f\} \\ \mathtt{cur}\, f = \{(z, g) \mid g = \{(x, y) \mid ((z, x), y) \in f\}\} \end{array}}$$

Haskell Curry

Haskell Brooks Curry (/ˈhæskəl/; September 12, 1900 – September 1, 1982) was an American mathematician and logician. Curry is best known for his work in combinatory logic; while the initial concept of combinatory logic was based on a single paper by Moses Schönfinkel,[1] much of the development was done by Curry. Curry is also known for Curry's paradox and the Curry–Howard correspondence. There are three programming languages named after him, Haskell, Brook and Curry, as well as the concept of *currying*, a

Haskell Brooks Curry	
	
Born	September 12, 1900 Millis, Massachusetts
Died	September 1, 1982 (aged 81) State College, Pennsylvania
Nationality	American
Alma mater	Harvard University
Known for	Combinatory logic Curry–Howard correspondence

For each function $f : Z \times X \to Y$ we get a commutative diagram in **Set**:

$$
\begin{tikzcd}
Y^X \times X \arrow[r, "\mathtt{app}"] & Y \\
Z \times X \arrow[u, "\mathtt{cur}\, f \times \mathtt{id}_X"] \arrow[ur, "f"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
(\mathtt{cur}\, f\, z, x) \arrow[r, mapsto] & \mathtt{cur}\, f\, z\, x = f(z,x) \\
(z,x) \arrow[u, mapsto] \arrow[ur, mapsto] &
\end{tikzcd}
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\end{tikzcd}
$$

Furthermore, if any function $g : Z \to Y^X$ also satisfies

$$
\begin{tikzcd}
Y^X \times X \arrow[r, "\mathtt{app}"] & Y \\
Z \times X \arrow[u, "g \times \mathtt{id}_X"] \arrow[ur, "f"']
\end{tikzcd}
$$

then $g = \mathtt{cur}\, f$, because of function extensionality…

Function Extensionality

Two functions $f, g \in Y^X$ are equal if (and only if) $\forall x \in X,\ f\,x = g\,x$.

This is true of the set-theoretic notion of function, because then

$$
\begin{array}{lc}
 & \{(x, f\,x) \mid x \in X\} = \{(x, g\,x) \mid x \in X\} \\
\text{i.e.} & \{(x, y) \mid (x, y) \in f\} = \{(x, y) \mid (x, y) \in g\} \\
\text{i.e.} & f = g
\end{array}
$$

(in other words it reduces to the extensionality property of sets: two sets are equal iff they have the same elements).

Exponential objects

Suppose a category $\mathbf{C}$ has binary products, that is, for every pair of $\mathbf{C}$-objects X and Y there is a product diagram $X \xleftarrow{\pi_1} X \times Y \xrightarrow{\pi_2} Y$.

Notation: given $f \in \mathbf{C}(X, X')$ and $f' \in \mathbf{C}(Y, Y')$, then $\boxed{f \times f' : X \times Y \to X' \times Y'}$ stands for $\langle f \circ \pi_1 \, , f' \circ \pi_2 \rangle$,
that is, the unique morphism $g \in \mathbf{C}(X \times Y, X' \times Y')$ satisfying

$\pi_1 \circ g = f \circ \pi_1$ and $\pi_2 \circ g = f' \circ \pi_2$.

Exponential objects

Suppose a category $\mathbf{C}$ has binary products.

An exponential for $\mathbf{C}$-objects X and Y is specified by

object Y^X + morphism $\mathtt{app} : Y^X \times X \to Y$

satisfying the universal property

for all $Z \in \mathbf{C}$ and $f \in \mathbf{C}(Z \times X, Y)$, there is a unique $g \in \mathbf{C}(Z, Y^X)$ such that

$$\begin{tikzcd} Y^X \times X \arrow[r, "\mathtt{app}"] & Y \\ Z \times X \arrow[u, "g \times \mathtt{id}_X"] \arrow[ur, "f"'] & \end{tikzcd}$$

commutes in $\mathbf{C}$.

Notation: we write $\boxed{\mathtt{cur}\, f}$ for the unique g such that $\mathtt{app} \circ (g \times \mathtt{id}_X) = f$.

Exponential objects

The universal property of $\mathtt{app} : Y^X \times X \to Y$ says that there is a bijection

$$
\begin{aligned}
\mathbf{C}(Z, Y^X) &\cong \mathbf{C}(Z \times X, Y) \\
g &\mapsto \mathtt{app} \circ (g \times \mathtt{id}_X) \\
\mathtt{cur}\, f &\leftarrow\!\shortmid f \\
\mathtt{app} \circ (\mathtt{cur}\, f \times \mathtt{id}_X) &= f \\
g &= \mathtt{cur}(\mathtt{app} \circ (g \times \mathtt{id}_X))
\end{aligned}
$$

Exponential objects

The universal property of $\mathtt{app} : Y^X \times X \to Y$ says that there is a bijection...

It also says that $(Y^X, \mathtt{app})$ is a terminal object in the following category:

- objects: (Z, f) where $f \in \mathbf{C}(Z \times X, Y)$
- morphisms $g : (Z, f) \to (Z', f')$ are $g \in \mathbf{C}(Z, Z')$ such that $f' \circ (g \times \mathtt{id}_X) = f$
- composition and identities as in $\mathbf{C}$.

So when they exist, exponential objects are unique up to (unique) isomorphism.

Cartesian closed category

Definition. $\mathbf{C}$ is a cartesian closed category (ccc) if it is a category with a terminal object, binary products and exponentials of any pair of objects.

This is a key concept for the semantics of lambda calculus and for the foundations of functional programming languages.

Notation: an exponential object Y^X is often written as $\boxed{X \to Y}$

Cartesian closed category

Definition. $\mathbf{C}$ is a cartesian closed category (ccc) if it is a category with a terminal object, binary products and exponentials of any pair of objects.

Examples:

- **Set** is a ccc — as we have seen.
- **Preord** is a ccc: we already saw that it has a terminal object and binary products; the exponential of $(P_1, \sqsubseteq_1)$ and $(P_2, \sqsubseteq_2)$ is $(P_1 \to P_2, \sqsubseteq)$ where

$$P_1 \to P_2 = \mathbf{Preord}((P_1, \sqsubseteq_1), (P_2, \sqsubseteq_2))$$
$$f \sqsubseteq g \;\Leftrightarrow\; \forall x \in P_1,\ f\,x \sqsubseteq_2 g\,x$$

(check that this is a pre-order and does give an exponential in **Preord**)