Lecture 4

Binary products

In a category $\mathbf{C}$, a product for objects $X, Y \in \mathbf{C}$ is a diagram $X \xleftarrow{\pi_1} P \xrightarrow{\pi_2} Y$ with the universal property:

For all $X \xleftarrow{f} Z \xrightarrow{g} Y$ in $\mathbf{C}$, there is a unique $\mathbf{C}$-morphism $h : Z \longrightarrow P$ such that the following diagram commutes in $\mathbf{C}$:

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[d, "h"] \arrow[dr, "g"] & \\ X & P \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & Y \end{tikzcd}$$

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For all $X \xleftarrow{f} Z \xrightarrow{g} Y$ in $\mathbf{C}$, there is a unique $\mathbf{C}$-morphism $h : Z \to P$ such that $f = \pi_1 \circ h$ and $g = \pi_2 \circ h$

So (P, π_1, π_2) is a terminal object in the category with

- objects: (Z, f, g) where $X \xleftarrow{f} Z \xrightarrow{g} Y$ in $\mathbf{C}$
- morphisms $h : (Z_1, f_1, g_1) \to (Z_2, f_2, g_2)$ are $h \in \mathbf{C}(Z_1, Z_2)$ such that $f_1 = f_2 \circ h$ and $g_1 = g_2 \circ h$
- composition and identities as in $\mathbf{C}$

So if it exists, the binary product of two objects in a category is unique up to (unique) isomophism.

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N.B. products of objects in a category do not always exist. For example in the category

$$\begin{tikzcd} 0 \arrow[loop left, "\mathtt{id}_0"] & & 1 \arrow[loop right, "\mathtt{id}_1"] \end{tikzcd}$$

two objects, no non-identity morphisms

the objects 0 and 1 do not have a product, because there is no diagram of the form $0 \leftarrow ? \to 1$ in this category.

Notation for binary products

Assuming $\mathbf{C}$ has binary products of objects, the product of $X, Y \in \mathbf{C}$ is written

$$\boxed{X \xleftarrow{\pi_1} X \times Y \xrightarrow{\pi_2} Y}$$

and given $X \xleftarrow{f} Z \xrightarrow{g} Y$, the unique $h : Z \to X \times Y$ with $\pi_1 \circ h = f$ and $\pi_2 \circ h = g$ is written

$$\boxed{\langle f\,, g\rangle : Z \to X \times Y}$$

Examples of products

In **Set**, category-theoretic products are given by the usual cartesian product of sets (set of all ordered pairs)

$$
\begin{aligned}
X \times Y &= \{(x, y) \mid x \in X \wedge y \in Y\} \\
\pi_1(x, y) &= x \\
\pi_2(x, y) &= y
\end{aligned}
$$

because…

Examples of products

In **Preord**, can take product of $(P_1, \sqsubseteq_1)$ and $(P_2, \sqsubseteq_2)$ to be

$$(P_1 \times P_2, \sqsubseteq)$$

product in **Set**

$$(x_1, x_2) \sqsubseteq (y_1, y_2) \Leftrightarrow$$
$$x_1 \sqsubseteq_1 y_1 \wedge x_2 \sqsubseteq_2 y_2$$

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The projection functions $P_1 \xleftarrow{\pi_1} P_1 \times P_2 \xrightarrow{\pi_2} P_2$ are monotone for this pre-order on $P_1 \times P_2$ and have the universal property needed for a product in **Preord** (check).

Examples of products

In **Mon**, can take product of $(M_1, \cdot_1, e_1)$ and $(M_2, \cdot_2, e_2)$ to be

$$(M_1 \times M_2, \ \cdot \ , (e_1, e_2))$$

product in **Set**

$$(x_1, x_2) \cdot (y_1, y_2) = (x_1 \cdot_1 y_1, x_2 \cdot_2 y_2)$$

Examples of products

In **Mon**, can take product of $(M_1, \cdot_1, e_1)$ and $(M_2, \cdot_2, e_2)$ to be

$$\big(\underbrace{M_1 \times M_2}_{\boxed{\text{product in } \mathbf{Set}}},\ \underbrace{\cdot}_{\boxed{\begin{array}{c}(x_1, x_2) \cdot (y_1, y_2) = \\ (x_1 \cdot_1 y_1, x_2 \cdot_2 y_2)\end{array}}}\ , (e_1, e_2)\big)$$

The projection functions $M_1 \xleftarrow{\pi_1} M_1 \times M_2 \xrightarrow{\pi_2} M_2$ are monoid morphisms for this monoid structure on $M_1 \times M_2$ and have the universal property needed for a product in **Mon** (check).

Examples of products

Recall that each pre-ordered set $(P, \sqsubseteq)$ determines a category $\mathbf{C}_P$.

Given $p, q \in P = \mathtt{obj}\, \mathbf{C}_P$, the product $p \times q$ (if it exists) is a greatest lower bound (or glb, or meet) for p and q in $(P, \sqsubseteq)$:

lower bound:

$p \times q \sqsubseteq p \;\; \wedge \;\; p \times q \sqsubseteq q$

greatest among all lower bounds:

$\forall r \in P,\;\; r \sqsubseteq p \;\; \wedge \;\; r \sqsubseteq q \;\; \Rightarrow \;\; r \sqsubseteq p \times q$

Notation: glbs are often written $\boxed{p \wedge q}$ or $\boxed{p \sqcap q}$

Duality

A binary coproduct of two objects in a category $\mathbf{C}$ is their product in the category $\mathbf{C}^{\mathrm{op}}$.

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Thus the coproduct of $X, Y \in \mathbf{C}$
if it exists,
is a diagram $X \xrightarrow{\mathtt{inl}} X + Y \xleftarrow{\mathtt{inr}} Y$
with the universal property:

$\forall\ (X \xrightarrow{f} Z \xleftarrow{g} Y),$

$\quad \exists!\ (X + Y \xrightarrow{h} Z),$

$\qquad f = h \circ \mathtt{inl} \wedge g = h \circ \mathtt{inr}$

Duality

A binary coproduct of two objects in a category $\mathbf{C}$ is their product in the category $\mathbf{C}^{\mathrm{op}}$.

E.g. in $\mathbf{Set}$, the coproduct of X and Y

$$X \xrightarrow{\mathtt{inl}} X + Y \xleftarrow{\mathtt{inr}} Y$$

is given by their disjoint union (tagged sum)

$$\begin{aligned} X + Y &= \{(0, x) \mid x \in X\} \cup \{(1, y) \mid y \in Y\} \\ \mathtt{inl}(x) &= (0, x) \\ \mathtt{inr}(y) &= (1, y) \end{aligned}$$

(prove this)