

## Lecture 2

Exercise Sheet 1 available (course web page)

Virtual Office hour –

Mondays 17:00-18:00

Zoom details on course Moodle page

# Recall

A **category**  $\mathbf{C}$  is specified by

- ▶ a set  $\text{obj } \mathbf{C}$  whose elements are called **C-objects**
- ▶ for each  $X, Y \in \text{obj } \mathbf{C}$ , a set  $\mathbf{C}(X, Y)$  whose elements are called **C-morphisms from  $X$  to  $Y$**
- ▶ a function assigning to each  $X \in \text{obj } \mathbf{C}$  an element  $\text{id}_X \in \mathbf{C}(X, X)$  called the **identity morphism** for the **C-object**  $X$
- ▶ a function assigning to each  $f \in \mathbf{C}(X, Y)$  and  $g \in \mathbf{C}(Y, Z)$  (where  $X, Y, Z \in \text{obj } \mathbf{C}$ ) an element  $g \circ f \in \mathbf{C}(X, Z)$  called the **composition** of **C-morphisms**  $f$  and  $g$  and satisfying **associativity** and **unity** properties.

# Example: category of pre-orders, **Preord**

- ▶ objects are sets  $P$  equipped with a **pre-order**  $\sqsubseteq$   
i.e. a binary relation on  $P$  that is

**reflexive:**  $\forall x \in P, x \sqsubseteq x$


**transitive:**  $\forall x, y, z \in P, x \sqsubseteq y \wedge y \sqsubseteq z \Rightarrow x \sqsubseteq z$

A **partial order** is a pre-order that is also

**anti-symmetric:**  $\forall x, y \in P, x \sqsubseteq y \wedge y \sqsubseteq x \Rightarrow x = y$

# Example: category of pre-orders, **Preord**

- ▶ objects are sets  $P$  equipped with a **pre-order**  $\sqsubseteq$
- ▶ morphisms: **Preord** $((P_1, \sqsubseteq_1), (P_2, \sqsubseteq_2)) \triangleq \{f \in \mathbf{Set}(P_1, P_2) \mid f \text{ is } \mathbf{monotone}\}$


$$\forall x, x' \in P_1, x \sqsubseteq_1 x' \Rightarrow f x \sqsubseteq_2 f x'$$

# Example: category of pre-orders, **Preord**

- ▶ objects are sets  $P$  equipped with a **pre-order**  $\_ \sqsubseteq \_$
- ▶ morphisms:  $\mathbf{Preord}((P_1, \sqsubseteq_1), (P_2, \sqsubseteq_2)) \triangleq \{f \in \mathbf{Set}(P_1, P_2) \mid f \text{ is monotone}\}$
- ▶ identities and composition: as for **Set**

Q: why is this well-defined?

A: because the set of monotone functions contains identity functions and is closed under composition.

# Example: category of pre-orders, **Preord**

- ▶ objects are sets  $P$  equipped with a **pre-order**  $\sqsubseteq$
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Pre- and partial orders are relevant to the denotational semantics of programming languages (among other things).

# Example: category of monoids, **Mon**

- ▶ objects are **monoids**  $(M, \cdot, e)$  — set  $M$  equipped with a binary operation  $\_ \cdot \_ : M \times M \rightarrow M$  which is **associative**  $\forall x, y, z \in M, x \cdot (y \cdot z) = (x \cdot y) \cdot z$  **has  $e$  as its unit**  $\forall x \in M, e \cdot x = x = x \cdot e$

CS-relevant example of a monoid:  $(\text{List } \Sigma, @, \text{nil})$  where

$\text{List } \Sigma$  = set of finite lists of elements of set  $\Sigma$   
 $@$  = list concatenation  
 $\text{nil } @ \ell = \ell$   
 $(a :: \ell) @ \ell' = a :: (\ell @ \ell')$   
 $\text{nil}$  = empty list

# Example: category of monoids, **Mon**

- ▶ objects are **monoids**  $(M, \cdot, e)$
- ▶ morphisms:  $\mathbf{Mon}((M_1, \cdot_1, e_1), (M_2, \cdot_2, e_2)) \triangleq$   
 $\{f \in \mathbf{Set}(M_1, M_2) \mid f e_1 = e_2 \wedge$   
 $\forall x, y \in M_1, f(x \cdot_1 y) = (f x) \cdot_2 (f y)\}$

It's common to denote a monoid  $(M, \cdot, e)$  just by its underlying set  $M$ , leaving  $_ \cdot _$  and  $e$  implicit (hence the same notation gets used for different instances of monoid operations).



# Example: category of monoids, **Mon**

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- ▶ identities and composition: as for **Set**

Q: why is this well-defined?

A: because the set of functions that are monoid morphisms contains identity functions and is closed under composition.

# Example: category of monoids, **Mon**

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 $\forall x, y \in M_1, f(x \cdot_1 y) = (f x) \cdot_2 (f y)\}$
- ▶ identities and composition: as for **Set**

Monoids are relevant to **automata theory** (among other things).

# Example: each pre-order determines a category

Given a pre-ordered set  $(P, \sqsubseteq)$ , we get a category  $\mathbf{C}_P$  by taking

▶ objects  $\text{obj } \mathbf{C}_P = P$

▶ morphisms  $\mathbf{C}_P(x, y) \triangleq \begin{cases} 1 & \text{if } x \sqsubseteq y \\ \emptyset & \text{if } x \not\sqsubseteq y \end{cases}$

(where  $1$  is some fixed one-element set and  $\emptyset$  is the empty set)

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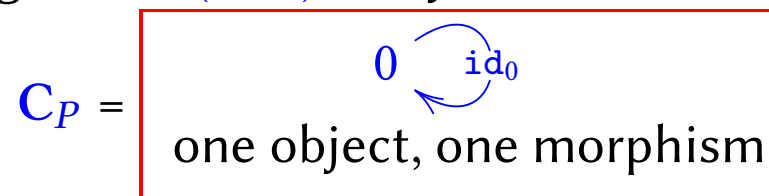
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E.g. when  $(P, \sqsubseteq)$  has just one element  $0$



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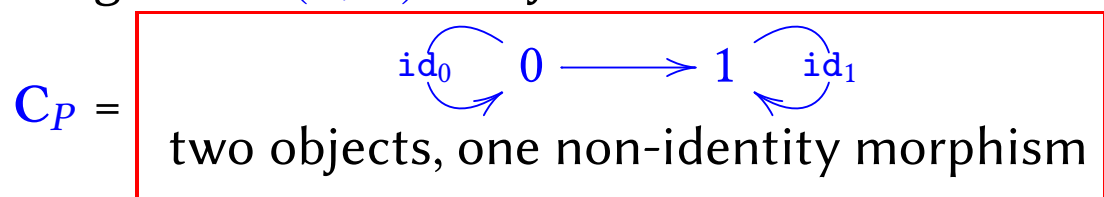
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▶ identity morphisms and composition are uniquely determined (why?)

E.g. when  $(P, \sqsubseteq)$  has just two elements  $0 \sqsubseteq 1$

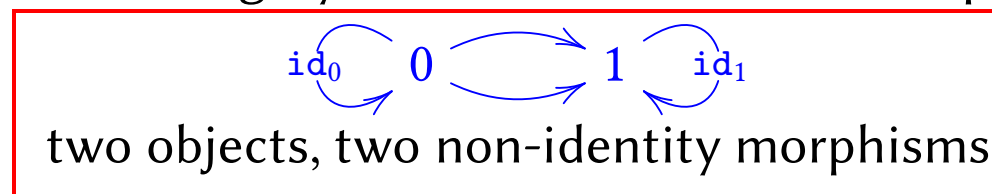


# Example: each pre-order determines a category

Given a pre-ordered set  $(P, \sqsubseteq)$ , we get a category  $\mathbf{C}_P$  by taking

- ▶ objects  $\text{obj } \mathbf{C}_P = P$
- ▶ morphisms  $\mathbf{C}_P(x, y) \triangleq \begin{cases} 1 & \text{if } x \sqsubseteq y \\ \emptyset & \text{if } x \not\sqsubseteq y \end{cases}$
- ▶ identity morphisms and composition are uniquely determined (**why?**)

Example of a finite category that does not arise from a pre-ordered set:



# Example: each monoid determines a category

Given a monoid  $(M, \cdot, e)$ ,  
we get a category  $\mathbf{C}_M$  by taking

- ▶ objects:  $\text{obj } \mathbf{C}_M = 1 = \{0\}$  (one-element set)
- ▶ morphisms:  $\mathbf{C}_M(0, 0) = M$
- ▶ identity morphism:  $\text{id}_0 = e \in M = \mathbf{C}_M(0, 0)$
- ▶ composition of  $f \in \mathbf{C}_M(0, 0)$  and  $g \in \mathbf{C}_M(0, 0)$  is  $g \cdot f \in M = \mathbf{C}_M(0, 0)$



# Definition of isomorphism

Let  $\mathbf{C}$  be a category. A  $\mathbf{C}$ -morphism  $f : X \rightarrow Y$  is an **isomorphism** if there is some  $g : Y \rightarrow X$  for which

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow \text{id}_X & \downarrow g \\ & & X & \xrightarrow{f} & Y \\ & & & \nearrow \text{id}_Y & \end{array}$$

is a commutative diagram.

# Definition of isomorphism

Let  $\mathbf{C}$  be a category. A  $\mathbf{C}$ -morphism  $f : X \rightarrow Y$  is an **isomorphism** if there is some  $g : Y \rightarrow X$  with  $g \circ f = \text{id}_X$  and  $f \circ g = \text{id}_Y$ .

▶ Such a  $g$  is uniquely determined by  $f$  (why?) and we write  $f^{-1}$  for it.

▶ Given  $X, Y \in \mathbf{C}$ , if such an  $f$  exists, we say the objects  $X$  and  $Y$  are **isomorphic** in  $\mathbf{C}$  and write

$$X \cong Y$$

(There may be many different  $f$  that witness the fact that  $X$  and  $Y$  are isomorphic.)

**Theorem.** A function  $f \in \mathbf{Set}(X, Y)$  is an isomorphism in the category  $\mathbf{Set}$  iff  $f$  is a bijection, that is

- ▶ **injective:**  $\forall x, x' \in X, f x = f x' \Rightarrow x = x'$
- ▶ **surjective:**  $\forall y \in Y, \exists x \in X, f x = y$

**Proof...**

**Theorem.** A function  $f \in \mathbf{Set}(X, Y)$  is an isomorphism in the category **Set** iff  $f$  is a bijection, that is

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Proof...

**Theorem.** A monoid morphism  $f \in \mathbf{Mon}((M_1, \cdot_1, e_1), (M_2, \cdot_2, e_2))$  is an isomorphism in the category **Mon** iff  $f \in \mathbf{Set}(M_1, M_2)$  is a bijection.

Proof...

Define **Poset** to be the category whose objects are **posets** = pre-ordered sets for which the pre-order is anti-symmetric, but is otherwise defined like the category **Preord** of pre-ordered sets.

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**Theorem.** A monotone function  $f \in \mathbf{Poset}((P_1, \sqsubseteq_1), (P_2, \sqsubseteq_2))$  is an isomorphism in the category **Poset** iff  $f \in \mathbf{Set}(P_1, P_2)$  is a surjective function satisfying

► **reflective:**  $\forall x, x' \in P_1, f x \sqsubseteq_2 f x' \Rightarrow x \sqsubseteq_1 x'$

Proof...

Define **Poset** to be the category whose objects are **posets** = pre-ordered sets for which the pre-order is anti-symmetric, but is otherwise defined like the category **Preord** of pre-ordered sets.

**Theorem.** A monotone function  $f \in \mathbf{Poset}((P_1, \sqsubseteq_1), (P_2, \sqsubseteq_2))$  is an isomorphism in the category **Poset** iff  $f \in \mathbf{Set}(P_1, P_2)$  is a surjective function satisfying

► **reflective**:  $\forall x, x' \in P_1, f x \sqsubseteq_2 f x' \Rightarrow x \sqsubseteq_1 x'$

**Proof...**

(Why does this characterisation not work for **Preord**?)