1. Recall (from Lecture 2) that a pre-ordered set \((P, \leq_P)\) determines a category \(C_P\) whose objects are the elements of \(P\) and whose morphism sets \(C_P(x, x')\) contain at most one element and do so if \(x \leq_P x'\). Note that given two pre-ordered sets \((P, \leq_P)\) and \((Q, \leq_Q)\), a functor \(F : C_P \to C_Q\) is the same thing as a monotone function from \((P, \leq_P)\) to \((Q, \leq_Q)\).

(a) Given two such functors \(F, G : C_P \to C_Q\), how many natural transformations are there from \(F\) to \(G\)?

(b) Given monotone functions \(F : C_P \to C_Q\) and \(G : C_Q \to C_P\), give a property of \(F\) and \(G\) which ensures that, regarding them as functors, \(G\) is right adjoint to \(F\).

2. Recall that \(\text{Preord}\) denotes the category of pre-ordered sets and monotone functions. For each set \(-\), let \(\left(\text{Pow}(-), \subseteq\right)\) be the set of all subsets of \(-\) partially ordered by inclusion (given \(A, A' \in \text{Pow}(-)\), \(A \subseteq A'\) means \(\forall x \in A, x \in A'\)). Given a function \(f : X \to Y\), let \(f^{-1} : \text{Pow}Y \to \text{Pow}X\) be the function that maps each subset \(B \subseteq Y\) to the subset \(f^{-1}B \subseteq X\) defined by \(f^{-1}B \triangleq \{x \in X \mid f(x) \in B\}\).

(a) Show that \(f^{-1}\) is a monotone function and hence gives a morphism \(\left(\text{Pow} Y, \subseteq\right) \to \left(\text{Pow} X, \subseteq\right)\) in \(\text{Preord}\).

(b) Regarding \(f^{-1}\) as a functor as in question (1), show that it has both left and right adjoints, given on objects by the following ‘generalized quantifiers’

\[
\exists f A \triangleq \{y \in Y \mid \exists x \in X, f(x) = y \land x \in A\}
\]

\[
\forall f A \triangleq \{y \in Y \mid \forall x \in X, f(x) = y \Rightarrow x \in A\}
\]

(for all \(A \in \text{Pow}(X)\). [Hint: use your answer to question 1b.]

3. A category \(C\) has pullbacks if for every pair of \(C\)-morphisms with a common codomain, \(Y \xrightarrow{f} X \xleftarrow{g} Z\), there is an object \(Y_f \times_g Z\) and morphisms \(p, q\) making the following diagram commute in \(C\) (that is, \(f \circ p = g \circ q\))

\[
\begin{array}{ccc}
Y & \xrightarrow{f} & Z \\
| & \downarrow{p} & \downarrow{q} \\
Y_f \times_g Z & \xrightarrow{g} & Z \\
| & \downarrow{f} & \downarrow{f} \\
Y & \xrightarrow{f} & X \\
\end{array}
\] (1)

and with the following universal property:

For all \(Y \xleftarrow{h} W \xrightarrow{k} Z\) in \(C\) with \(f \circ h = g \circ k\), there is a unique morphism \(t \in C(W, Y_f \times_g Z)\)
satisfying \( p \circ \ell = h \) and \( q \circ \ell = k \)

\[\begin{array}{ccc}
W & \xrightarrow{k} & Z \\
\downarrow & & \downarrow \\
Y \times_Z Z & \xrightarrow{q} & Z \\
\downarrow & & \downarrow \\
Y & \xrightarrow{f} & X
\end{array}\]

(a) Show that \( C \) has pullbacks iff for all \( x \in \text{obj} \ C \) the slice category \( C/X \) (defined in Exercise Sheet 4, question 2) has binary products.

(b) Show that if \( C \) has a terminal object and pullbacks, then it has binary products.

(c) Suppose \( C \) has pullbacks. Given \( f \in C(Y, X) \), show that the mapping

\[ f^* : Y \times_Z Z \rightarrow Z \]

is the object part of a functor \( f^* : C/X \rightarrow C/Y \) between slice categories.

(d) Show that the functor \( f^* \) in part (c) always has a left adjoint \( f_! : C/Y \rightarrow C/X \), which on objects sends \( (W, h) \in \text{obj}(C/Y) \) to \( f!(W, h) = (W, f \circ h) \in \text{obj}(C/X) \).

4. Suppose \((T, \eta, \mu)\) is a monad on a category \( C \) (see Lecture 16). Thus \( T : C \rightarrow C \) is a functor and \( \eta : \text{id}_C \rightarrow T \) and \( \mu : T \circ T \rightarrow T \) are natural transformations satisfying \( \mu \circ T \eta = \text{id}_T = \mu \circ \eta \) and \( \mu \circ \mu_T = \mu \circ T \mu \) (see Exercise Sheet 5, question 5 for the notation being used in those equations). The Kleisli category \( C_T \) of the monad is defined as follows. It has the same objects as \( C \); we will write \( X \) for the object of \( C_T \) corresponding to an object \( x \in \text{obj} \ C \). Given \( X, Y \in \text{obj} \ C \), the set of morphisms in the Kleisli category from \( X \) to \( Y \) is defined to be \( C_T(FX, FY) = C(X, TY) \).

(a) Complete the definition of \( C_T \) by giving the definition of identity morphisms and composition satisfying the usual associativity and unity properties.

(b) Show that the mapping \( X \in \text{obj} \ C \mapsto FX \in \text{obj} \ C_T \) extends to a functor \( F : C \rightarrow C_T \).

(c) Show that the functor \( F \) has a right adjoint \( G : C_T \rightarrow C \).

(d) Show that the monad associated with the adjunction \( F \vdash G \) (see Lecture 16) is \((T, \eta, \mu)\).