Type Systems

Lecture 9: Classical Logic

Neel Krishnaswami
University of Cambridge

Where We Are

We have seen the Curry Howard correspondence:

- Intuitionistic propositional logic $\longleftrightarrow$ Simply-typed lambda calculus
- Second-order intuitionistic logic $\longleftrightarrow$ Polymorphic lambda calculus

We have seen effectful programs:

- State
- I/O
- Monads

But what about:

- Control operators (eg, exceptions, `goto`, etc)
- Classical logic

A Review of Intuitionistic Propositional Logic

$$\frac{P \in \Psi}{\Psi \vdash P \text{ true}}\ \textsc{Hyp} \qquad \frac{}{\Psi \vdash \top \text{ true}}\ \top\text{I}$$

$$\frac{\Psi \vdash P \text{ true} \qquad \Psi \vdash Q \text{ true}}{\Psi \vdash P \wedge Q \text{ true}}\ \wedge\text{I} \qquad \frac{\Psi \vdash P_1 \wedge P_2 \text{ true}}{\Psi \vdash P_i \text{ true}}\ \wedge\text{E}_i$$

$$\frac{\Psi, P \vdash Q \text{ true}}{\Psi \vdash P \supset Q \text{ true}}\ \supset\text{I} \qquad \frac{\Psi \vdash P \supset Q \text{ true} \qquad \Psi \vdash P \text{ true}}{\Psi \vdash Q \text{ true}}\ \supset\text{E}$$

Disjunction and Falsehood

$$\frac{\Psi \vdash P \text{ true}}{\Psi \vdash P \vee Q \text{ true}} \vee\mathsf{I}_1 \qquad \frac{\Psi \vdash Q \text{ true}}{\Psi \vdash P \vee Q \text{ true}} \vee\mathsf{I}_2$$

$$\frac{\Psi \vdash P \vee Q \text{ true} \qquad \Psi, P \vdash R \text{ true} \qquad \Psi, Q \vdash R \text{ true}}{\Psi \vdash R \text{ true}} \vee\mathsf{E}$$

(no intro for $\bot$) $$\frac{\Psi \vdash \bot \text{ true}}{\Psi \vdash R \text{ true}} \bot\mathsf{E}$$

Intuitionistic Propositional Logic

- Key judgement: $\Psi \vdash R$ true
  - "If everything in Ψ is true, then R is true"
- Negation $\neg P$ is a derived notion
  - Definition: $\neg P = P \to \bot$
  - "Not P" means "P implies false"
  - To *refute* P means to give a *proof* that P implies false

What if we treat refutations as a first-class notion?

A Calculus of Truth and Falsehood

Propositions	A	$::=$	$\top \mid A \wedge B \mid \bot \mid A \vee B \mid \neg A$
True contexts	Γ	$::=$	$\cdot \mid \Gamma, A$
False contexts	Δ	$::=$	$\cdot \mid \Delta, A$

Proofs	$\Gamma; \Delta \vdash A$ true	If Γ is true and Δ is false, A is true
Refutations	$\Gamma; \Delta \vdash A$ false	If Γ is true and Δ is false, A is false
Contradictions	$\Gamma; \Delta \vdash$ contr	Γ and Δ contradict one another

- $\neg A$ is primitive (no implication $A \to B$)
- Eventually, we'll encode it as $\neg A \vee B$

Proofs

$$\frac{A \in \Gamma}{\Gamma; \Delta \vdash A \ \text{true}} \ \textsc{Hyp}$$

(No rule for $\bot$)

$$\frac{}{\Gamma; \Delta \vdash \top \ \text{true}} \ \top\text{P}$$

$$\frac{\Gamma; \Delta \vdash A \ \text{true} \qquad \Gamma; \Delta \vdash B \ \text{true}}{\Gamma; \Delta \vdash A \wedge B \ \text{true}} \ \wedge\text{P}$$

$$\frac{\Gamma; \Delta \vdash A \ \text{true}}{\Gamma; \Delta \vdash A \vee B \ \text{true}} \ \vee\text{P}_1 \qquad \frac{\Gamma; \Delta \vdash B \ \text{true}}{\Gamma; \Delta \vdash A \vee B \ \text{true}} \ \vee\text{P}_2$$

$$\frac{\Gamma; \Delta \vdash A \ \text{false}}{\Gamma; \Delta \vdash \neg A \ \text{true}} \ \neg\text{P}$$

Refutations

$$\frac{A \in \Delta}{\Gamma; \Delta \vdash A \text{ false}} \textsc{Hyp}$$

(No rule for $\top$)

$$\frac{}{\Gamma; \Delta \vdash \bot \text{ false}} \bot R$$

$$\frac{\Gamma; \Delta \vdash A \text{ false} \qquad \Gamma; \Delta \vdash B \text{ false}}{\Gamma; \Delta \vdash A \vee B \text{ false}} \vee R$$

$$\frac{\Gamma; \Delta \vdash A \text{ false}}{\Gamma; \Delta \vdash A \wedge B \text{ false}} \wedge R_1 \qquad \frac{\Gamma; \Delta \vdash B \text{ false}}{\Gamma; \Delta \vdash A \wedge B \text{ false}} \wedge R_2$$

$$\frac{\Gamma; \Delta \vdash A \text{ true}}{\Gamma; \Delta \vdash \neg A \text{ false}} \neg R$$

75% of the Way to Classical Logic

Connective	To Prove	To Refute
$\top$	Do nothing	Impossible!
$A \wedge B$	Prove A *and* prove B	Refute A *or* refute B
$\bot$	Impossible!	Do nothing
$A \vee B$	Prove A *or* prove B	Refute A *and* refute B
$\neg A$	Refute A	Prove A

Something We Can Prove: A entails $\neg\neg A$

$$\dfrac{\dfrac{\dfrac{}{A; \cdot \vdash A \ \text{true}}\ \textsc{Hyp}}{A; \cdot \vdash \neg A \ \text{false}}\ \neg\text{R}}{A; \cdot \vdash \neg\neg A \ \text{true}}\ \neg\text{P}$$

Something We Cannot Prove: $\neg\neg A$ entails A

$$\frac{???}{\neg\neg A; \cdot \vdash A \;\; \text{true}}$$

- There is no rule that applies in this case
- Proofs and refutations are mutually recursive
- But we have no way to use assumptions!

Something Else We Cannot Prove: $A \wedge B$ entails A

$$\frac{???}{A \wedge B; \cdot \vdash A \;\; \text{true}}$$

- This is intuitionistically valid: $\lambda x : A \times B.\, \mathsf{fst}\, x$
- But it's not derivable here
- Again, we can't use hypotheses nontrivially

A Bold Assumption

- Proofs and refutations are perfectly symmetrical
- This suggests the following idea:
    1. To refute A means to give direct evidence it is false
    2. This is also how we prove $\neg A$
    3. If we show a contradiction from assuming A is false, we have proved it
    4. If we can show a contradiction from assuming A is true, we have refuted it

$$\frac{\Gamma; \Delta, A \vdash \text{contr}}{\Gamma; \Delta \vdash A \;\; \text{true}} \qquad \frac{\Gamma, A; \Delta \vdash \text{contr}}{\Gamma; \Delta \vdash A \; \text{false}}$$

Contradictions

$$\frac{\Gamma; \Delta \vdash A \ \ \text{true} \qquad \Gamma; \Delta \vdash A \ \text{false}}{\Gamma; \Delta \vdash \text{contr}} \ \textsc{Contr}$$

- A contradiction arises when A has a proof *and* a refutation

Double Negation Elimination

$$
\dfrac{\dfrac{}{\neg\neg A; A \vdash \neg\neg A \ \text{true}} \qquad \dfrac{\dfrac{\dfrac{}{\neg\neg A; A \vdash A \ \text{false}}}{\neg\neg A; A \vdash \neg A \ \text{true}}}{\neg\neg A; A \vdash \neg\neg A \ \text{false}}}{\dfrac{\neg\neg A; A \vdash \text{contr}}{\neg\neg A; \cdot \vdash A \ \text{true}}}
$$

Projections: $A \wedge B$ entails A

$$
\dfrac{\dfrac{\dfrac{}{A \wedge B; A \vdash A \wedge B \ \text{true}} \qquad \dfrac{\dfrac{}{A \wedge B; A \vdash A \ \text{false}}}{A \wedge B; A \vdash A \wedge B \ \text{false}}}{A \wedge B; A \vdash \text{contr}}}{A \wedge B; \cdot \vdash A \ \text{true}}
$$

Projections: $A \vee B$ false entails A false

$$\dfrac{\dfrac{}{A; A \vee B \vdash A \vee B \text{ false}} \qquad \dfrac{\dfrac{}{A; A \vee B \vdash A \ \text{ true}}}{A; A \vee B \vdash A \vee B \ \text{ true}}}{\dfrac{A; A \vee B \vdash \text{contr}}{\cdot; A \vee B \vdash A \text{ false}}}$$

The Excluded Middle

$$
\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\cdot; A \vee \neg A \vdash A \text{ false}}}{\cdot; A \vee \neg A \vdash \neg A \ \text{ true}}}{\cdot; A \vee \neg A \vdash A \vee \neg A \ \text{ true}} \qquad \dfrac{}{\cdot; A \vee \neg A \vdash A \vee \neg A \text{ false}}}{\dfrac{\cdot; A \vee \neg A \vdash \text{contr}}{\cdot; \cdot \vdash A \vee \neg A \ \text{ true}}}
$$

Proof (and Refutation) Terms

Propositions	A	$::=$	$\top \mid A \wedge B \mid \bot \mid A \vee B \mid \neg A$
True contexts	Γ	$::=$	$\cdot \mid \Gamma, x : A$
False contexts	Δ	$::=$	$\cdot \mid \Delta, u : A$
Values	e	$::=$	$\langle\rangle \mid \langle e, e'\rangle \mid \mathsf{L}\, e \mid \mathsf{R}\, e \mid \mathsf{not}(k)$
		$\mid$	$\mu u : A.\, c$
Continuations	k	$::=$	$[] \mid [k, k'] \mid \mathsf{fst}\, k \mid \mathsf{snd}\, k \mid \mathsf{not}(e)$
		$\mid$	$\mu x : A.\, c$
Contradictions	c	$::=$	$\langle e \mid_A k\rangle$

Expressions — Proof Terms

$$\frac{x : A \in \Gamma}{\Gamma; \Delta \vdash x : A \text{ true}} \textsc{Hyp}$$

(No rule for $\bot$)

$$\frac{}{\Gamma; \Delta \vdash \langle\rangle : \top \text{ true}} \top\mathrm{P}$$

$$\frac{\Gamma; \Delta \vdash e : A \text{ true} \qquad \Gamma; \Delta \vdash e' : B \text{ true}}{\Gamma; \Delta \vdash \left\langle e, e' \right\rangle : A \wedge B \text{ true}} \wedge\mathrm{P}$$

$$\frac{\Gamma; \Delta \vdash e : A \text{ true}}{\Gamma; \Delta \vdash \mathsf{L}\, e : A \vee B \text{ true}} \vee\mathrm{P}_1 \qquad \frac{\Gamma; \Delta \vdash e : B \text{ true}}{\Gamma; \Delta \vdash \mathsf{R}\, e : A \vee B \text{ true}} \vee\mathrm{P}_2$$

$$\frac{\Gamma; \Delta \vdash k : A \text{ false}}{\Gamma; \Delta \vdash \mathrm{not}(k) : \neg A \text{ true}} \neg\mathrm{P}$$

Continuations — Refutation Terms

$$\frac{x : A \in \Delta}{\Gamma; \Delta \vdash x : A \text{ false}} \textsc{Hyp}$$

(No rule for $\top$)

$$\frac{}{\Gamma; \Delta \vdash [] : \bot \text{ false}} \bot R$$

$$\frac{\Gamma; \Delta \vdash k : A \text{ false} \qquad \Gamma; \Delta \vdash k' : B \text{ false}}{\Gamma; \Delta \vdash \left[k, k'\right] : A \vee B \text{ false}} \vee R$$

$$\frac{\Gamma; \Delta \vdash k : A \text{ false}}{\Gamma; \Delta \vdash \mathsf{fst}\, k : A \wedge B \text{ false}} \wedge R_1 \qquad \frac{\Gamma; \Delta \vdash k : B \text{ false}}{\Gamma; \Delta \vdash \mathsf{snd}\, k : A \wedge B \text{ false}} \wedge R_2$$

$$\frac{\Gamma; \Delta \vdash e : A \text{ true}}{\Gamma; \Delta \vdash \mathsf{not}(e) : \neg A \text{ false}} \neg R$$

Contradictions

$$\frac{\Gamma; \Delta \vdash e : A \text{ true} \qquad \Gamma; \Delta \vdash k : A \text{ false}}{\Gamma; \Delta \vdash \langle e \mid_A k \rangle \text{ contr}} \textsc{Contr}$$

$$\frac{\Gamma; \Delta, u : A \vdash c \text{ contr}}{\Gamma; \Delta \vdash \mu u : A.\, c : A \text{ true}} \qquad \frac{\Gamma, x : A; \Delta \vdash c \text{ contr}}{\Gamma; \Delta \vdash \mu x : A.\, c : A \text{ false}}$$

Operational Semantics

$$
\begin{array}{lcl}
\langle \langle e_1, e_2 \rangle \mid_{A \wedge B} \mathsf{fst}\, k \rangle & \mapsto & \langle e_1 \mid_A k \rangle \\
\langle \langle e_1, e_2 \rangle \mid_{A \wedge B} \mathsf{snd}\, k \rangle & \mapsto & \langle e_2 \mid_B k \rangle \\
\langle \mathsf{L}\, e \mid_{A \vee B} [k_1, k_2] \rangle & \mapsto & \langle e \mid_A k_1 \rangle \\
\langle \mathsf{R}\, e \mid_{A \vee B} [k_1, k_2] \rangle & \mapsto & \langle e \mid_B k_2 \rangle \\
\langle \mathsf{not}(k) \mid_{\neg A} \mathsf{not}(e) \rangle & \mapsto & \langle e \mid_A k \rangle
\end{array}
$$

$$
\begin{array}{lcl}
\langle \mu u : A.\, c \mid_A k \rangle & \mapsto & [k/u]c \\
\langle e \mid_A \mu x : A.\, c \rangle & \mapsto & [e/x]c
\end{array}
$$

A Bit of Non-Determinism

$$\langle \mu u : A.\, c \mid_A \mu x : A.\, c' \rangle \quad \mapsto \quad ?$$

- Two rules apply!
- Different choices of priority correspond to *evaluation order*
- Similar situation in the simply-typed lambda calculus
- The STLC is *confluent*, so evaluation order doesn't matter
- But in the classical case, evaluation order matters a lot!

Metatheory: Substitution

- If $\Gamma; \Delta \vdash e : A$ true then
  1. If $\Gamma, x : A; \Delta \vdash e' : C$ true then $\Gamma; \Delta \vdash [e/x]e' : C$ true.
  2. If $\Gamma, x : A; \Delta \vdash k : C$ false then $\Gamma; \Delta \vdash [e/x]k : C$ false.
  3. If $\Gamma, x : A; \Delta \vdash c$ contr then $\Gamma; \Delta \vdash [e/x]c$ contr.
- If $\Gamma; \Delta \vdash k : A$ false then
  1. If $\Gamma; \Delta, u : A \vdash e' : C$ true then $\Gamma; \Delta \vdash [k/u]e' : C$ true.
  2. If $\Gamma; \Delta, x : A \vdash k' : C$ false then $\Gamma; \Delta \vdash [k/u]k' : C$ false.
  3. If $\Gamma; \Delta, u : A \vdash c$ contr then $\Gamma; \Delta \vdash [k/u]c$ contr.

- We *also* need to prove weakening and exchange!
- Because there are 2 kinds of assumptions, and 3 kinds of judgement, there are $2 \times 3 = 6$ lemmas!

What Is This For?

- We have introduced a proof theory for classical logic
- Expected tautologies and metatheory holds...
- ...but it looks totally different from STLC?
- Computationally, this is a calculus for *stack machines*
- Related to *continuation passing style* (next lecture!)

Questions

1. Show that $\neg A \vee B, A; \cdot \vdash B$ true is derivable
2. Show that $\neg(\neg A \wedge \neg B); \cdot \vdash A \vee B$ true is derivable
3. Prove substitution for values (you may assume exchange and weakening hold).