Introduction to Probability: Homework 1

1. [Ross, Chapter 1, Problem 10] In how many ways can 8 people be seated in a row if
    a. there are no restrictions on the seating arrangement?
    b. persons A and B must sit next to each other?
    c. there are 4 men and 4 women and no 2 men or 2 women can sit next to each other?
    d. there are 5 men and they must sit next to one another?
    e. there are 4 married couples and each couple must sit together?

2. [Ross, Chapter 1, Problem 31] If 8 identical blackboards are to be divided among 4 schools, how many divisions are possible? How many if each school must receive at least 1 blackboard?
    Hint: the blackboards are identical, it only matters how many boards each school gets.

3. [Ross, Chapter 1, Theoretical Exercise 8] Prove that

$$\binom{n+m}{r} = \binom{n}{0}\binom{m}{r} + \binom{n}{1}\binom{m}{r-1} + \cdots + \binom{n}{r}\binom{m}{0}$$

    Hint: while you could do this by induction, instead prove it by interpreting the left-hand side and the right-hand side of the equality combinatorially.

4. [Ross, Chapter 2, Problem 3] Two dice are thrown. Let E be the event that the sum of the dice is odd, let F be the event that at least one of the dice lands on 1, and let G be the event that the sum is 5. Describe the events EF, EUF, FG, EF, and EFG. Assuming fair dice, what are the probabilities of these events?

5. [Ross, Chapter 2, Self-test exercise 14] Prove Boole's inequality:

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} \mathbb{P}(A_i).$$

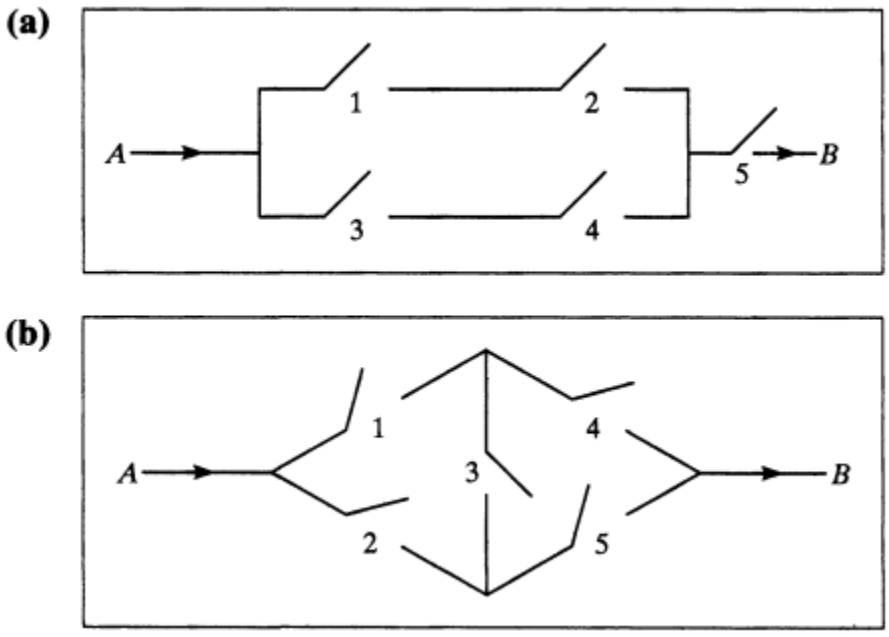
6. [Ross, Chapter 2, Problem 29] An um contains n white and m black balls, where n and m are positive numbers.
    a. If two balls are randomly withdrawn, what is the probability that they are the same color?

b. If a ball is randomly withdrawn and then replaced what is the probability that the withdrawn balls are the same color?
c. Show that the probability in part (b) is always larger than the one in part (a).

7. [Ross, Chapter 3, Problem 18] A total of 46 percent of the voters in a certain city classify themselves as Independents, whereas 30 percent classify themselves as Liberals and 24 percent say that they are Conservatives. In a recent local election, 35 percent of the Independents, 62 percent of the Liberals, and 58 percent of the Conservatives voted. A voter is chosen at random. Given that this person voted in the local election, what is the probability that he or she is

a. an Independent?
b. a Liberal?
c. a Conservative?
d. What percent of voters participated in the local election?

8. [Ross, Chapter 3, Problem 66] The probability of the closing of the ith relay in the circuits shown in the figure is given by *Pi. i* = 1, 2, 3, 4, 5. If all relays function independently, what is the probability that a current flows between A and B for the respective circuits? **Hint for (b):** condition on whether relay 3 closes.



9. [Ross, Chapter 3, Theoretical Exercise 22] As a simplified model for weather forecasting, suppose that the weather (either wet or dry) tomorrow will be the same as the weather today with probability p. Show that if the weather is dry on January 1, then
Pn, the probability that it will be dry n days later, satisfies:

$$P_n = (2p - 1)P_{n-1} + (1 - p) \qquad n \geq 1$$
$$P_0 = 1$$

Use this formula to prove by induction:

$$P_n = \frac{1}{2} + \frac{1}{2}(2p - 1)^n \qquad n \geq 0$$

10. [Ross, Chapter 3, Theoretical Exercise 4] A ball is in any one of n boxes and is in the ith box with probability Pi. If the ball is in box i, a search of that box will uncover it with probability ai. Show that the conditional probability that the ball is in box j, given that a search of box i did not uncover it, is

$$\frac{P_j}{1 - \alpha_i P_i} \quad \text{if } j \neq i$$
$$\frac{(1 - \alpha_i)P_i}{1 - \alpha_i P_i} \quad \text{if } j = i$$

11. [Ross, Chapter 4, Problem 11] A salesman has scheduled two appointments to sell encyclopedias. His first appointment will lead to a sale with probability .3, and his second will lead independently to a sale with probability .6. Any sale made is equally likely to be either for the deluxe model, which costs $1000, or the standard model, which costs $500.

a. Determine the probability mass function of X, the total dollar value of all sales.
b. Compute the expected revenue.
c. Compute the variance and standard deviation of this distribution.

12. Suppose X and Y are two discrete random variables. Show that:
Var(X + Y) = Var(X) + Var(Y) + 2(E(X)E(Y) − E(XY))

13. Suppose X and Y are independent random variables with cumulative distribution functions F_X and F_Y. Compute the distribution functions of e^X, min(X,Y) and max(X,Y).