Additional Exercises for Introduction to Probability (Lectures 1-6)

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Lectures 1, 2 (Introduction to Probability)

1. We have a hash table that has 100 buckets. We add to the table two arbitrary strings that are independently hashed. How many possible ways are there for the strings to be stored in the table?

2. A license plate has 6 places, where the first three are upper case letters A-Z, and the last three places are numeric 0-9. How many such 6-place license plates are possible?

3. Consider a hash table with 100 buckets. 950 strings are hashed and added to the table. a) Is it possible that a bucket in the table contains no entries? b) Is it guaranteed that at least one bucket in the table contains at least two entries? c) Is it guaranteed that at least one bucket in the table contains at least 10 entries? d) Is it guaranteed that at least one bucket in the table contains at least 11 entries?

4. In a town of 8000 people, how many ways are there to choose 2 people from a group of distinct 12 people?

5. How many ways are there to select 3 books from a set of 6?

6. Old iPhone passcodes were 4-digit. If we can see a fingerprint on the screen of 3 digits (so 1 digit must use used twice), how many distinct passcodes are possible? What is there were only fingerprints only on 2 digits?

7. How many distinct bit strings can be formed from three 0's and two 1's?

8. A company X has 13 new servers that they would like to assign to 3 datacentres, where datacentre A, B and C have 6, 4 and 3 empty server racks, respectively. How many different divisions of the serves are possible?

Lectures 3, 4 (Random variables, probability mass function, expectation, expectation properties, variance, discrete distributions)

1. You are playing a card game that uses four standard decks of cards. There are 208 card in total. Each deck has 52 cards (13 values with 4 suits each). Cards are only distinguishable based in their suit and value, not which deck they came from.

a) In how many distinct ways can the cards be ordered?
b) You will be dealt the first two cards from the four decks of cards. Cards with values 10, Jack, Queen, King and Ace are considered "good" cards. What is the probability of getting two "good" cards?
c) Over the course of several rounds you observe 100 cards played. Out of the cards played only 15 were "good" cards. You are dealt the next two cards. What is the probability of getting two "good" cards now? You may assume that previously seen cards are not re-dealt.

2. n people go to a party and drop off their hats to a hat-check person. When the party is over, a different hat-check person is on duty, and returns the n hats randomly back to each person. Let X be the random variable representing the number of people who get their own hat back.
a) For $n = 3$, find $E[X]$ by first computing the probability mass function p_X, and then applying the definition of expectation.
b) Find a general formula for $E[X]$, for any positive integer n.

3. Four 6-sided dice are rolled. The dice are fair, so each one has equal probability of producing a value in $\{1, 2, 3, 4, 5, 6\}$. Let X = the minimum of the four values rolled. (It is fine if more than one of the dice has the minimal value.)
a) What is $P[X \geqslant k]$ as a function of k?
b) What is $E[X]$?
c) Let T be the sum of the values rolled on the four dice. Let S be the sum of the largest three values on the four dice. In other words, $S = T - X$. What is $E[S]$?

4. The lottery works like this: 100 balls numbered 0-99 are placed in an urn, and 5 balls are withdrawn, giving an ordered sequence of 5 numbers. Once balls are drawn they are not replaced before the next are drawn. Citizens buy tickets with 5 numbers of their choosing. The jackpot is awarded for matching all 5 numbers in the right order.
a) When drawing a lottery, how many possible outcomes are there?
b) You decide to play 5 different numbers. You buy several tickets, one for each of permutations of these 5 numbers. How many tickets do you need to buy? What is your chance of winning the jackpot?
c) You can also win a prize if you guess all 5 numbers, but in incorrect order. What is the probability that any ticket could win this prize? What is your probability of winning this prize?

5. A four sided die has sides numbered $1 - 4$. You roll two such dice. Let X be the sum of the two dice.
a) What are the possible values of X?
b) What is the probability mass function of X?
c) What is the expectation of X, $E[X]$?

6. Let X be a randon variable with possible values of 0 and 1. Let $P[X = 0] = 2 \cdot P[X = 1]$ and $E[X] = \frac{1}{3}$.
a) What is the probability mass function of X?
b) What is the variance of of X, $V[X]$?

Lecture 5 (More discrete distributions: Poisson, Geometric, Negative)

1. Cambridge Tigers Korfball (CTK) team has a probability 0.7 to win in a home game, and probability 0.5 to win in an away game. All games are independent. In a season, there are

35 home and 35 away games.
a) What is the probability that CTK win exactly 20 home games?
b) What is the probability that the first home win for CTK is on the fourth home game?
c) CTK plays in a 3-game series in a pattern home-away-home. What is the probability that CTK wins two out of these three games?
d) A new korfball team joins the league and only has probability of 0.05 of winning each game regardless of whether it is home or away. What is the approximation of the probability that this new team wins w season games?

2. A take-away has a footfall of 10 customers per hour. Let us model with by a Poisson process.
a) What is the probability that at least 3 customers come to a take-away over a house of an hour?
b) For how many hours must the take-away be open for the expected number of customers visiting in that time period to be 100? What is the probability that 0 customer come to the take-away during that time period?
c) Suppose that starting at any particular moment in time, the amount of time we must wait for the next customer to come is a continuous random variable with probability density function $f(x)$ where x is measured in hours:

$$f(x) = \begin{cases} 10e^{-10x} & \text{if } x \geqslant 0, \\ 0 & \text{if } x < 0. \end{cases}$$

What is the probability that we must wait more than 6 minutes (which is 0.1 hours) for the next customer to come to take-away?

Lecture 6 (Continuous random variables)

1. Let us toss a fair coin 100 times. Let X be the number of heads. What is the probability that $43 \leqslant X \leqslant 57$? Using continuity correction, approximate the binomial distribution by normal distribution.

2. Two types of cars, A and B, are being tested for fuel consumption. On test, there are 10 cars of type A and 10 cars of type B, and the consumption of fuel by all 20 vehicles is independent. Let A_x be the number of litres of fuel by car x of type A consumed in the test drive. Let B_y be the number of litres of fuel by car y of type B consumed in the same test drive. Let $A_x \sim Pois(4)$ for $1 \leqslant x \leqslant 10$ and $B_y \sim N(5, 3)$ for $1 \leqslant y \leqslant 10$. For every test drive we pick (probabilistically) cars that we will monitor for the amount of fuel consumed and generate the statistics for them. For every test drive a car has (independently) a 0.2 probability of being monitored. Let T be the total amount of fuel (in litres) consumed by the monitored cars.
a) Compute $E[T]$.
b) On a particular test drive, let there be 3 type A cars being monitored (no type B). What is $P[T \geqslant 20]$ on that test drive?
c) Now, on another test drive, let there be 3 type B cars being monitored (no type A). What is $P[T \geqslant 20]$ on that test drive?

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