Foundations of Computer Science

A list application: making change

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An Application: Making Change

Till has **unlimited supply of coins**, for certain coin values

List of coins in till given in **descending order**

Larger coins preferred (tried first)

```
In[1]: let rec change till amt =
         match till, amt with
         | _, 0          -> []
         | [], _         -> raise (Failure "no more coins!")
         | c::till,amt -> if amt < c then change till amt
                          else c :: change (c::till) (amt - c)

Out[1]: val change : int list -> int -> int list = <fun>
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The recursion terminates when amt $= 0$

Tries the **largest coin first**

The algorithm is **greedy**, and it **can fail**!

list of possible coin values

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Out[2]: val till : int list = [50; 20; 10; 5; 2; 1]
In[3]: change till 43
50  20 (amt=23)  20 (amt=3)  10  5  2 (amt=1)  1 (amt=0)
Out[3]: - : int list = [20; 20; 2; 1]
In[4]: let till = [5; 2]
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In[5]: change till 16
5 (amt=11)  5 (amt=6)  5 (amt=1)  2  ? amt≠0, till=[]
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? amt $\neq$ 0, till =[]

(**Disclaimer**: This is rather hard!)

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In[6]: let rec change till amt =
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generates all possible solutions

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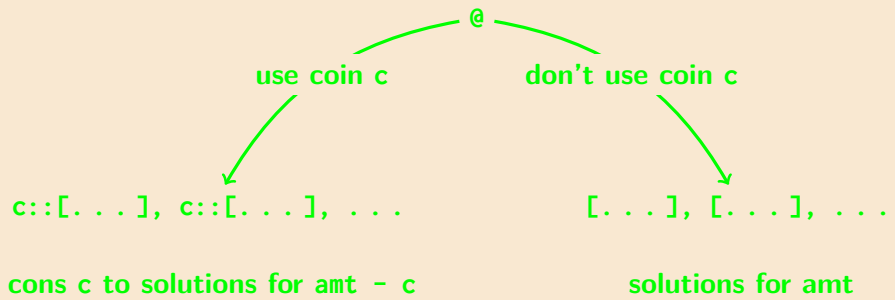

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In[7]: let till = [5; 3; 2];;
       change till 6

Out[7]: - : int list list = [[3; 3]; [2; 2; 2]]

In[8]: let till = [5; 2];;
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ALL Ways of Making Change

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Out[9]: val change : int list -> int -> int list ->
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We've added **another accumulating parameter**!

Repeatedly improving simple code is called **stepwise refinement**.

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use coin

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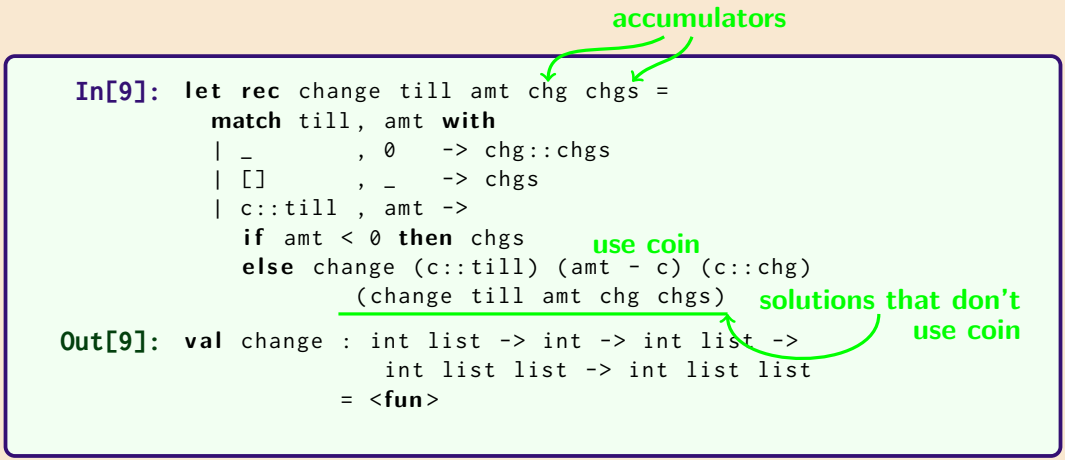
ALL Ways of Making Change — Faster!



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