Foundations of Computer Science

The basics of lists

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2020–2021

Warm-Up

Question 1: What does this return?

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In[1]: 3 + -0.2

  Out: Error: This expression has type float
       but an expression was expected of type int
       Line 1, characters 2-3:
       Hint: Did you mean to use '+.'?
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Question 2: What is the **complexity** of matrix addition for a square matrix of size n?

Question 3: What do we call a function whose computation does not nest?

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Iterative or tail-recursive

Lists

A list is a **finite sequence** of elements

The elements may have **any type**

All elements must have **same** type

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 In[2]: [3; 5; 9]
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 In[6]: it @ [2; 10]
Out[6]: - : int list = [3; 5; 9; 2; 10]
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reverse

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 In[7]: List.rev [(1, "one"); (2, "two")]
Out[7]: - : (int * string) list
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The List Primitives

We build a list using **two primitives**:

$$[\,]$$
$$::$$

Example: the list $[3;\ 5;\ 9]$ is constructed as follows:

$$\begin{aligned} 9 \ :: \ [] &= [9] \\ 5 \ :: \ [9] &= [5;\ 9] \\ 3 \ :: \ [5;\ 9] &= [3;\ 5;\ 9] \end{aligned}$$

The List Primitives

Two kinds of list

[]	is the empty list
x :: l	is the list with head x and tail l

List notation

$$[x_1;\ x_2;\ \ldots;\ x_n] \quad \equiv \quad \underbrace{x_1}_{\text{head}} :: \underbrace{(x_2 :: \ \cdots (x_n :: []))}_{\text{tail}}$$

The List Primitives

Internally: linked structure

$$\begin{tikzcd}
:: \arrow[r] \arrow[d] & :: \arrow[r] \arrow[d] & :: \arrow[r] \arrow[d] & :: \arrow[r] \arrow[d] & {[]} \\
1 & 3 & 5 & 9 &
\end{tikzcd}$$

Note that :: is an $O(1)$ operation

Taking a list's head or tail takes **constant time**

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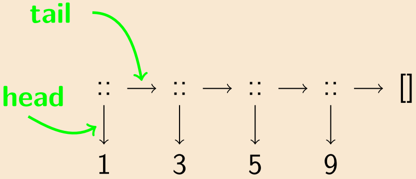

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 In[8]: let rec up_to m n =
          if m > n then []
          else m :: up_to (m + 1) n

Out[8]: val up_to : int -> int -> int list = <fun>

 In[9]: up_to 2 5

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Getting at the Head and Tail

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 In[10]: let hd (x::_) = x
         Warning 8: this pattern-matching is not
         exhaustive.
         Here is an example of a case that is not
         matched:
         []

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 In[11]: List.tl [7; 6; 5]

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Pattern-matching

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 In[12]: let null = function
 1st case ⟶| [] -> true
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Getting at the Head and Tail

Note: all three functions are **polymorphic**:

`val null : 'a list -> bool`	is a list empty?
`val hd : 'a list -> 'a`	head of a non-empty list
`val tl : 'a list -> 'a list`	tail of a non-empty list

Computing the Length of a List

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         | [] -> 0
         | _ :: xs -> 1 + nlength xs

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What is the **time and space complexity** of this function?

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Efficiently Computing the Length of a List

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 In[14]: let rec addlen n = function
         | [] -> n
         | _::xs -> addlen (n + 1) xs

Out[14]: val addlen : int -> 'a list -> int = <fun>

 In[15]: let length xs = addlen 0 xs

Out[15]: val length : 'a list -> int = <fun>
```

$$
\begin{aligned}
\text{length } [3;\ 5;\ 9] &\Rightarrow \text{addlen } 0\ [3;\ 5;\ 9]\\
&\Rightarrow \text{addlen } 1\ [5;\ 9]\\
&\Rightarrow \text{addlen } 2\ [9]\\
&\Rightarrow \text{addlen } 3\ []
\end{aligned}
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What is the **time and space complexity** of this function?