Discrete Differential Geometry

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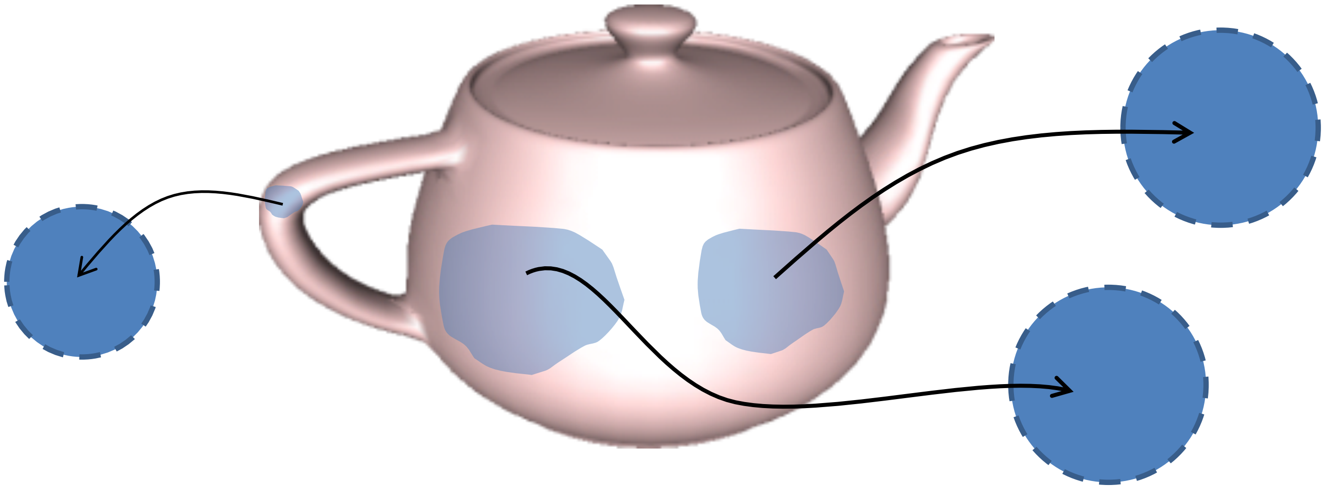
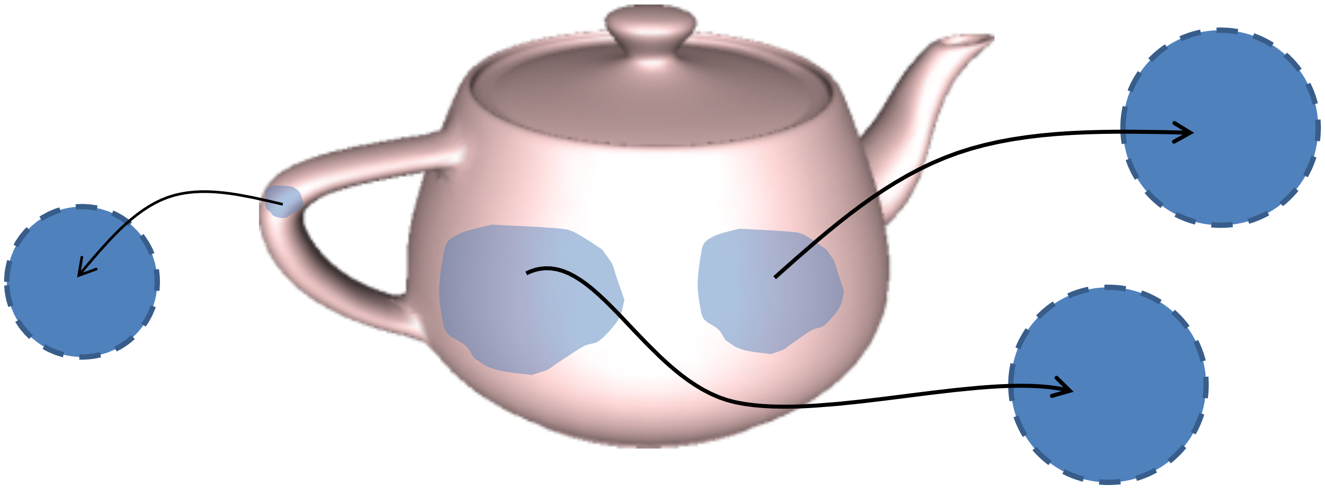
Manifolds

- A surface is a closed **2-manifold** if it is locally homeomorphic to a disk everywhere



Manifolds

- For every point x in M, there is an **open** ball $B_x(r)$ of radius r centered at x such that $M \cap B_x(r)$ is homeomorphic to an open disk



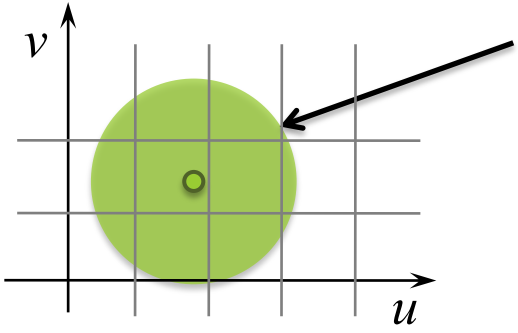
Manifolds with Boundary

- Each boundary point is homeomorphic to a half-disk

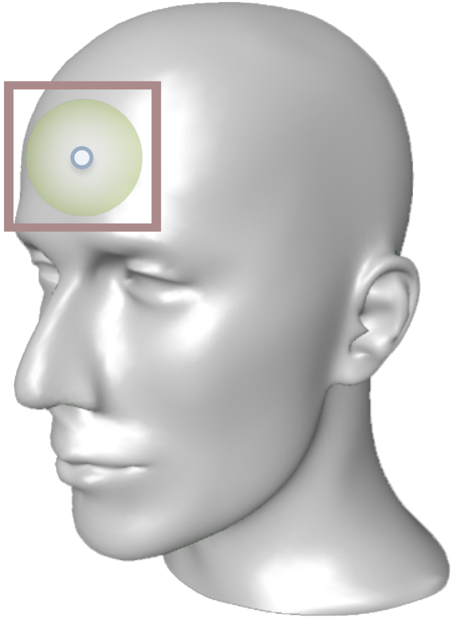
Differential Geometry Basics

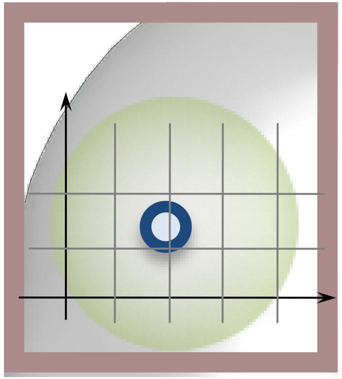
Things that can be discovered by local observation

continuous 1-1 mapping

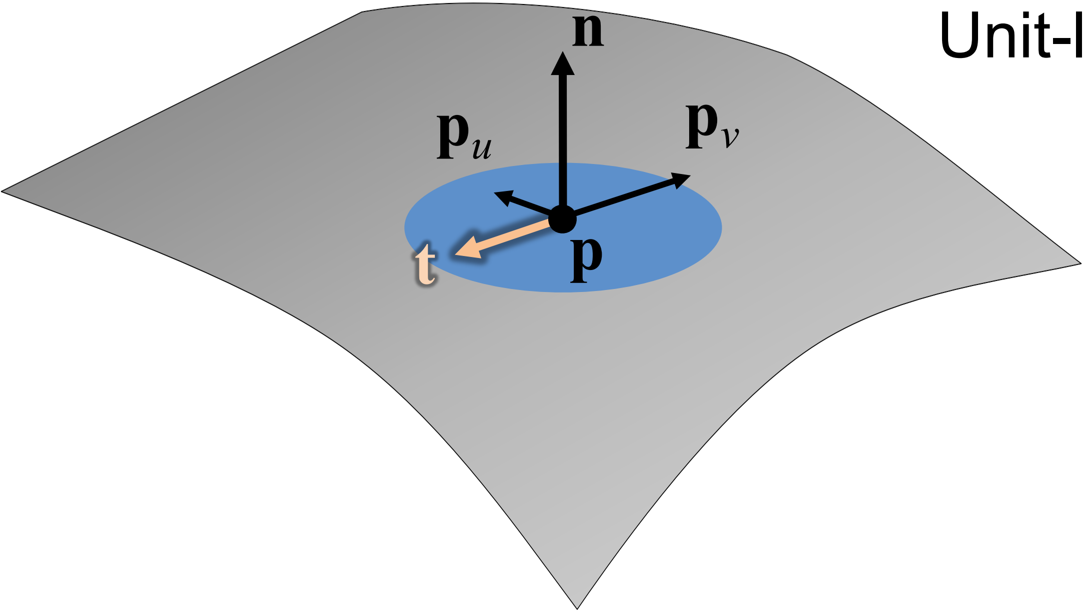


$$\frac{\partial}{\partial u} \quad \frac{\partial}{\partial v} \quad \frac{\partial^2}{\partial^2 u} \quad \frac{\partial^2}{\partial u \partial v} \cdots$$





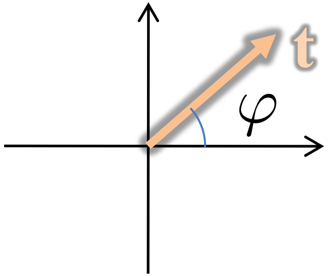
Normal Curvature



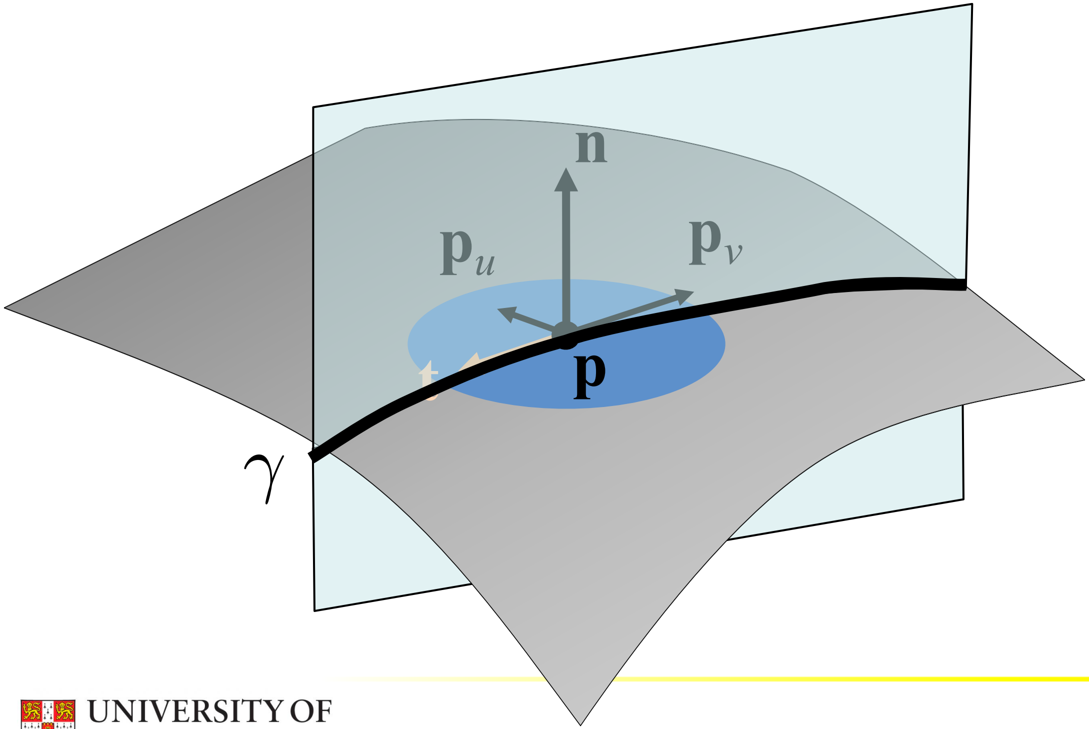
Unit-length $\mathbf{t}$ in the tangent plane

If $\mathbf{p}_u$ and $\mathbf{p}_v$ are orthogonal:

$$\mathbf{t} = \cos\varphi \frac{\mathbf{p}_u}{\|\mathbf{p}_u\|} + \sin\varphi \frac{\mathbf{p}_v}{\|\mathbf{p}_v\|}$$



Normal Curvature



The curve γ is the intersection of the surface with the plane through $\mathbf{n}$ and $\mathbf{t}$.

Normal curvature:

$$\kappa_n(\varphi) = \kappa_n(\gamma(\mathbf{p}))$$

Surface Curvatures

- Principal curvatures
  - Minimal curvature $\kappa_1 = \kappa_{\min} = \min_{\varphi} \kappa_n(\varphi)$
  - Maximal curvature $\kappa_2 = \kappa_{\max} = \max_{\varphi} \kappa_n(\varphi)$
- Mean curvature $H = \frac{\kappa_1 + \kappa_2}{2} = \frac{1}{2\pi} \int_0^{2\pi} \kappa_n(\varphi) d\varphi$
- Gaussian curvature $K = \kappa_1 \cdot \kappa_2$

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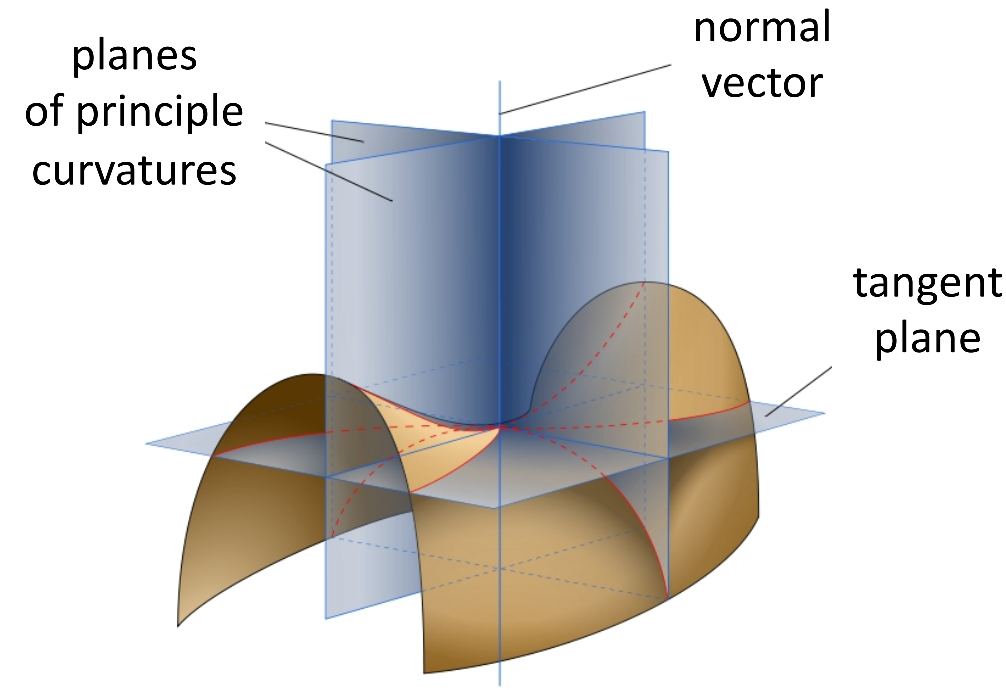
Principle Directions



Euler's Theorem:
Planes of principal curvature are **orthogonal** and independent of parameterization.

$$\kappa_n(\varphi) = \kappa_1 \cos^2 \varphi + \kappa_2 \sin^2 \varphi$$

$$\varphi = \text{angle with } \mathbf{t}_1$$

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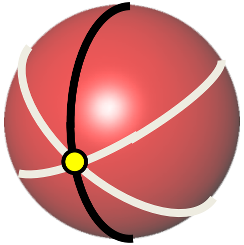
Local Shape by Curvatures

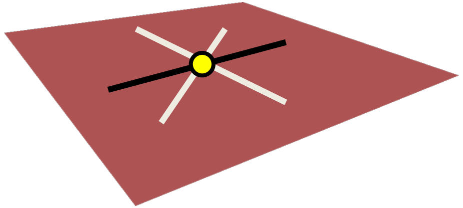
Isotropic:
all directions are
principal directions

spherical (umbilical)



$$K > 0, \kappa_1 = \kappa_2$$

planar

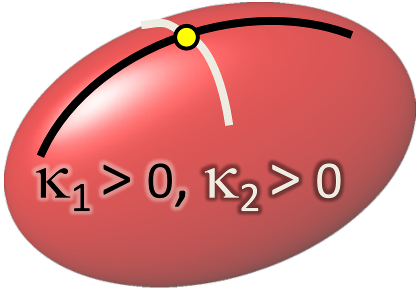


$$K = 0$$

Local Shape by Curvatures

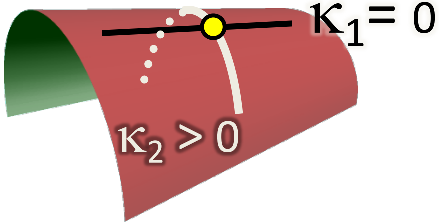
Anisotropic:
2 distinct principal directions

elliptic



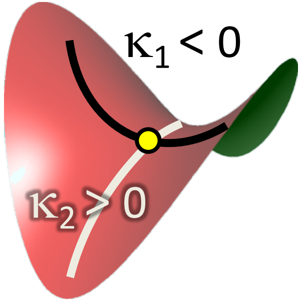
$$K > 0$$

parabolic



$$K = 0$$

hyperbolic

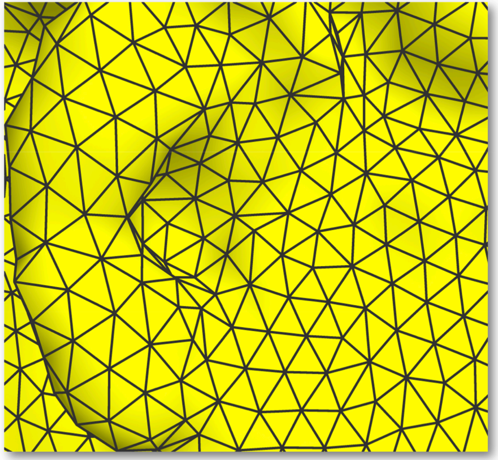


$$K < 0$$

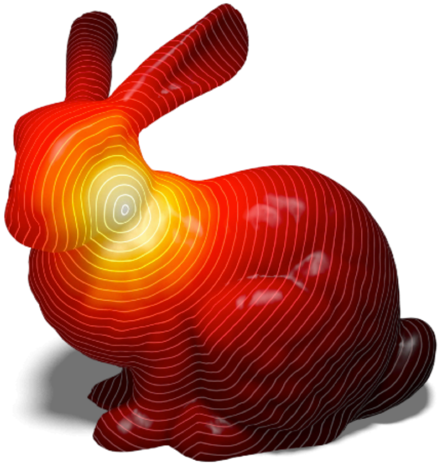
Discrete Differential Geometry

- Approximate surface normal and curvature via

Local surface approximation



Global: discrete Laplace-Beltrami



Laplace Operator

$$f : \mathbb{R}^3 \to \mathbb{R}$$

$$\Delta f : \mathbb{R}^3 \to \mathbb{R}$$

Laplace operator

gradient operator

2nd partial derivatives

$$\Delta f = \mathrm{div} \nabla f$$

function in Euclidean space

divergence operator

Cartesian coordinates

Laplace Operator

$$f : \mathbb{R}^3 \to \mathbb{R} \qquad \Delta f : \mathbb{R}^3 \to \mathbb{R}$$

$$\Delta f = \mathrm{div} \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \ldots$$

$$\mathrm{grad} f = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \qquad \mathrm{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

Laplace-Beltrami Operator

- Extension to manifold surfaces

$$f : \mathcal{M} \to \mathbb{R}$$
$$\Delta f : \mathcal{M} \to \mathbb{R}$$

Laplace-
Beltrami

gradient
operator

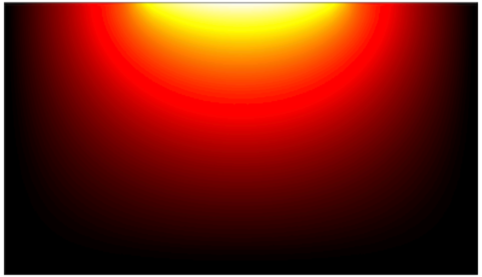
$$\Delta_{\mathcal{M}} f = \mathrm{div}_{\mathcal{M}} \nabla_{\mathcal{M}} f$$

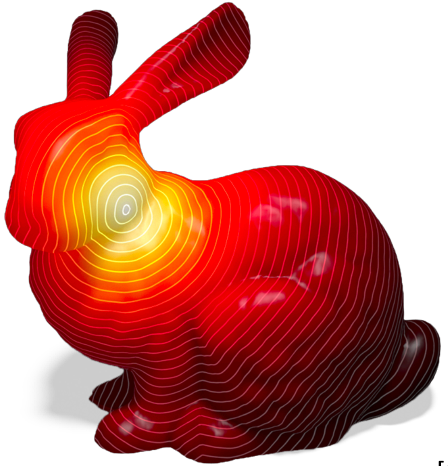
function on
surface *M*

divergence
operator

Laplace-Beltrami Operator

- Example: heat equation





[Crane et al. 2013]

$$\Delta f = 0$$

$$\text{s.t. } f|_{\partial\Omega} = f_0$$

$$\Delta_{\mathcal{M}} f = 0$$

$$\text{s.t. } \textit{boundary conditions}$$

Laplace-Beltrami Operator

- Apply to coordinate function

$$f(x, y, z) = x \quad \mathbf{p} = (x, y, z)$$

Laplace-Beltrami

gradient operator

mean curvature

$$\Delta_{\mathcal{M}} \mathbf{p} = \mathrm{div}_{\mathcal{M}} \nabla_{\mathcal{M}} \mathbf{p} = -2H\mathbf{n} \ \in \mathbb{R}^3$$

function on surface *M*

divergence operator

unit surface normal

Laplace-Beltrami Operator

- Apply to coordinate function

$$f(x, y, z) = x \quad \mathbf{p} = (x, y, z)$$

Laplace-Beltrami

mean curvature

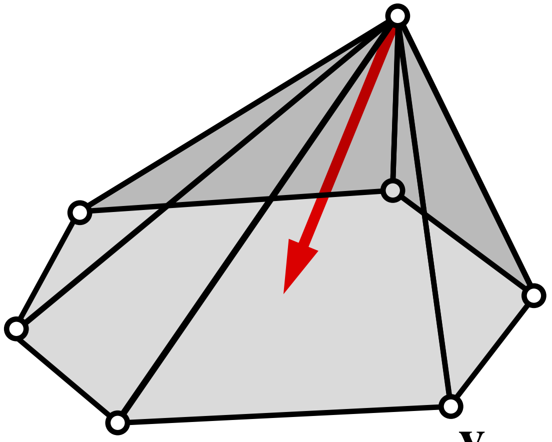
$$\Delta_{\mathcal{M}} \mathbf{p} = -2H\mathbf{n}$$

function on surface *M*

unit surface normal

Discrete Laplace-Beltrami

$$\Delta_{\mathcal{M}} \mathbf{p} = -2H\mathbf{n}$$

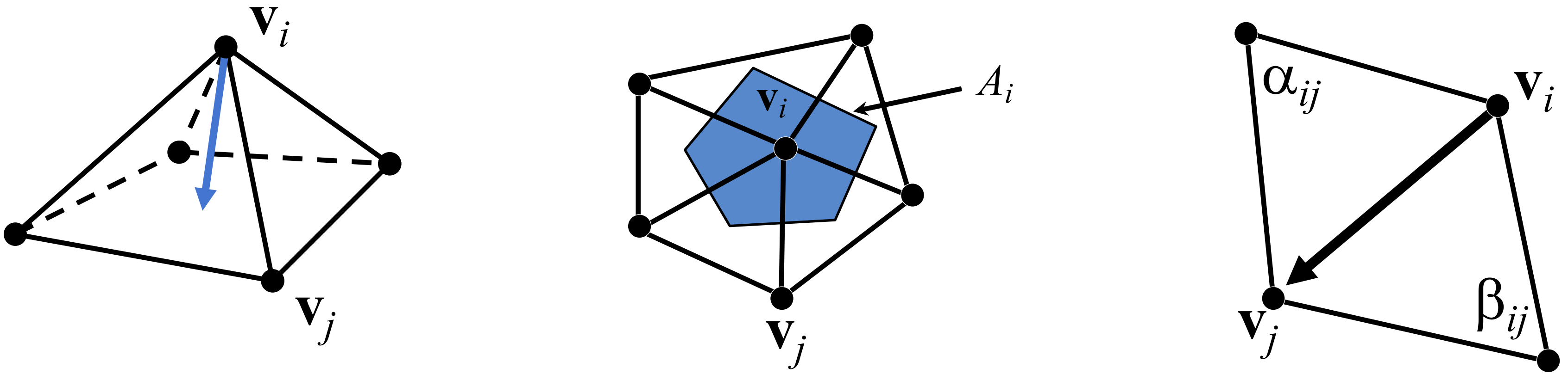


$$L_u(\mathbf{v}_i) = \frac{1}{|\mathcal{N}(i)|} \sum_{j \in \mathcal{N}(i)} (\mathbf{v}_j - \mathbf{v}_i)$$

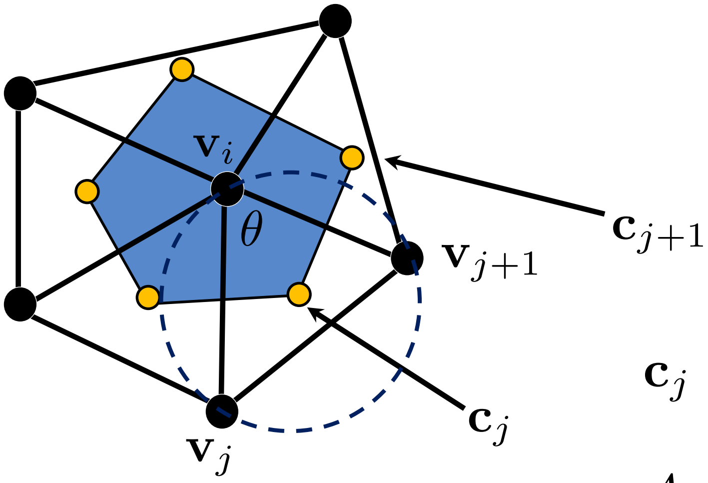
$$= \left(\frac{1}{|\mathcal{N}(i)|} \sum_{j \in \mathcal{N}(i)} \mathbf{v}_j \right) - \mathbf{v}_i$$

Discrete Laplace-Beltrami

$$L_c(\mathbf{v}_i) = \frac{1}{A_i} \sum_{j \in \mathcal{N}(i)} \frac{1}{2} (\cot \alpha_{ij} + \cot \beta_{ij})(\mathbf{v}_j - \mathbf{v}_i)$$



Discrete Laplace-Beltrami



$$\mathbf{c}_j = \begin{cases} \text{circumcenter of } \triangle(\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_{j+1}) & \text{if } \theta < \pi/2 \\ \text{midpoint of edge } (\mathbf{v}_j, \mathbf{v}_{j+1}) & \text{if } \theta \geq \pi/2 \end{cases}$$

$$A_i = \sum_j \text{Area}\left(\triangle(\mathbf{v}_i, \mathbf{c}_j, \mathbf{c}_{j+1})\right)$$

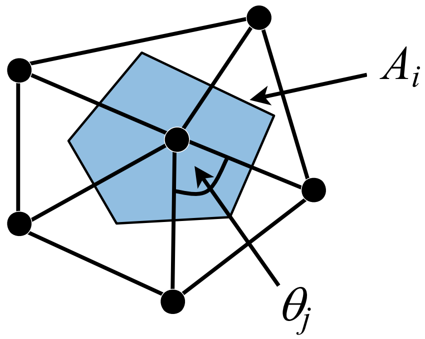
Relation to Normal & Curvature

- Mean curvature (sign according to normal)

$$|H(\mathbf{v}_i)| = \|L_c(\mathbf{v}_i)\|/2$$

- Gaussian curvature

$$K(\mathbf{v}_i) = \frac{1}{A_i}(2\pi - \sum_j \theta_j)$$

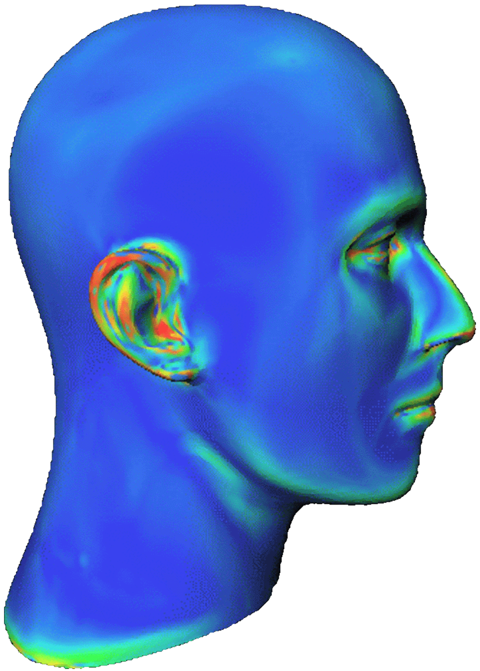


- Principal curvatures

$$\kappa_1 = H - \sqrt{H^2 - K} \qquad \kappa_2 = H + \sqrt{H^2 - K}$$

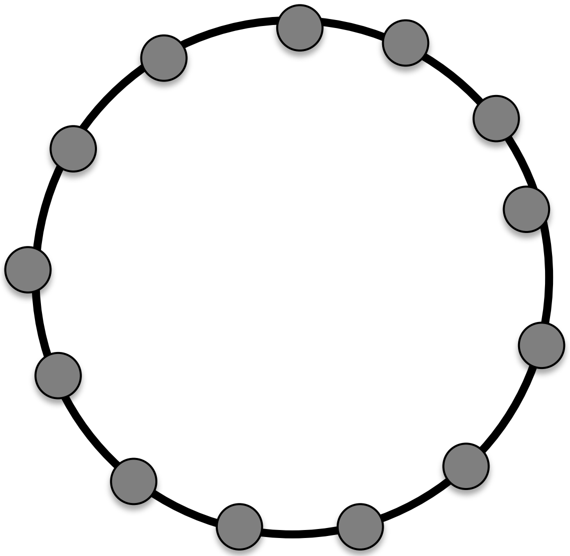
Discrete Curvatures

Mean Curvature



Discrete Laplace-Beltrami

- Extension to graphs and point clouds



$$h_t(x_i, x_j) = e^{-d(x_i, x_j)/t}$$

$$= e^{-\|\mathbf{x}_i - \mathbf{x}_j\|^2/t}$$

$$L_g(x_i) = \frac{1}{\sum_{j=1}^{n} h_t(x_i, x_j)} \sum_{j=1}^{n} h_t(x_i, x_j)(x_j - x_i)$$

Discrete Laplace-Beltrami

- Extension to graphs and point clouds