

Discrete Mathematics

Supervision 4

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4. On induction

4.1. Basic exercises

1. Prove that for all natural numbers $n \geq 3$, if n distinct points on a circle are joined in consecutive order by straight lines, then the interior angles of the resulting polygon add up to $180 \cdot (n - 2)$ degrees.
2. Prove that, for any positive integer n , a $2^n \times 2^n$ square grid with any one square removed can be tiled with L-shaped pieces consisting of 3 squares.

4.2. Core exercises

1. Establish the following:

(a) For all positive integers m and n ,

$$(2^n - 1) \cdot \sum_{i=0}^{m-1} 2^{i \cdot n} = 2^{m \cdot n} - 1$$

(b) Suppose k is a positive integer that is not prime. Then $2^k - 1$ is not prime.

2. Prove that

$$\forall n \in \mathbb{N}. \forall x \in \mathbb{R}. x \geq -1 \implies (1 + x)^n \geq 1 + n \cdot x$$

3. Recall that the Fibonacci numbers F_n for $n \in \mathbb{N}$ are defined recursively by $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_n + F_{n+1}$ for $n \in \mathbb{N}$.

a) Prove Cassini's Identity: For all $n \in \mathbb{N}$,

$$F_n \cdot F_{n+2} = F_{n+1}^2 + (-1)^{n+1}$$

b) Prove that for all natural numbers k and n ,

$$F_{n+k+1} = F_{n+1} \cdot F_{k+1} + F_n \cdot F_k$$

c) Deduce that $F_n \mid F_{l \cdot n}$ for all natural numbers n and l .

d) Prove that $\text{gcd}(F_{n+2}, F_{n+1})$ terminates with output 1 in n steps for all positive integers n .

e) Deduce also that:

(i) for all positive integers $n < m$, $\text{gcd}(F_m, F_n) = \text{gcd}(F_{m-n}, F_n)$,

and hence that:

(ii) for all positive integers m and n , $\text{gcd}(F_m, F_n) = F_{\text{gcd}(m, n)}$.

- f) Show that for all positive integers m and n , $(F_m \cdot F_n) \mid F_{m \cdot n}$ if $\gcd(m, n) = 1$.
- g) Conjecture and prove theorems concerning the following sums for any natural number n :
- (i) $\sum_{i=0}^n F_{2 \cdot i}$
 - (ii) $\sum_{i=0}^n F_{2 \cdot i + 1}$
 - (iii) $\sum_{i=0}^n F_i$

4.3. Optional exercises

- Recall the **gcd0** function from §3.3.3. Use the Principle of Mathematical Induction from basis 2 to formally establish the following correctness property of the algorithm:

For all natural numbers $l \geq 2$, we have that for all positive integers m, n , if $m + n \leq l$ then **gcd0**(m, n) terminates.

- The set of *univariate polynomials* (over the rationals) on a variable x is defined as that of arithmetic expressions equal to those of the form $\sum_{i=0}^n a_i \cdot x^i$, for some $n \in \mathbb{N}$ and some coefficients $a_0, a_1, \dots, a_n \in \mathbb{Q}$.

- (a) Show that if $p(x)$ and $q(x)$ are polynomials then so are $p(x) + q(x)$ and $p(x) \cdot q(x)$.
- (b) Deduce as a corollary that, for all $a, b \in \mathbb{Q}$, the linear combination $a \cdot p(x) + b \cdot q(x)$ of two polynomials $p(x)$ and $q(x)$ is a polynomial.
- (c) Show that there exists a polynomial $p_2(x)$ such that $p_2(n) = \sum_{i=0}^n i^2 = 0^2 + 1^2 + \dots + n^2$ for every $n \in \mathbb{N}$.¹

Hint: Note that for every $n \in \mathbb{N}$,

$$(n+1)^3 = \sum_{i=0}^n (i+1)^3 - \sum_{i=0}^n i^3$$

- (d) Show that, for every $k \in \mathbb{N}$, there exists a polynomial $p_k(x)$ such that, for all $n \in \mathbb{N}$, $p_k(n) = \sum_{i=0}^n i^k = 0^k + 1^k + \dots + n^k$.

Hint: Generalise the hint above, and the similar identity

$$(n+1)^2 = \sum_{i=0}^n (i+1)^2 - \sum_{i=0}^n i^2$$

¹Chapter 2.5 of *Concrete Mathematics* by R.L. Graham, D.E. Knuth and O. Patashnik looks at this in great detail.