Discrete Mathematics

Supervision 1

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1. On proofs

1.1. Basic exercises

The main aim is to practice the analysis and understanding of mathematical statements (e.g. by isolating the different components of composite statements), and exercise the art of presenting a logical argument in the form of a clear proof (e.g. by following proof strategies and patterns).

Prove or disprove the following statements.

1. Suppose n is a natural number larger than 2, and n is not a prime number. Then $2 \cdot n + 13$ is not a prime number.
2. If $x^2 + y = 13$ and $y \neq 4$ then $x \neq 3$.
3. For an integer n, n^2 is even if and only if n is even.
4. For all real numbers x and y there is a real number z such that $x + z = y - z$.
5. For all integers x and y there is an integer z such that $x + z = y - z$.
6. The addition of two rational numbers is a rational number.
7. For every real number x, if $x \neq 2$ then there is a unique real number y such that $2 \cdot y/(y+1) = x$.
8. For all integers m and n, if $m \cdot n$ is even, then either m is even or n is even.

1.2. Core exercises

Having practised how to analyse and understand basic mathematical statements and clearly present their proofs, the aim is to get familiar with the basics of divisibility.

1. Characterise those integers d and n such that:
   a) $0 \mid n$
   b) $d \mid 0$
2. Let k, m, n be integers with k positive. Show that:
$$(k \cdot m) \mid (k \cdot n) \iff m \mid n$$
3. Prove or disprove that: For all natural numbers n, $2 \mid 2^n$.
4. Show that for all integers l, m, n,
$$l \mid m \ \wedge \ m \mid n \Longrightarrow l \mid n$$
5. Find a counterexample to the statement: For all positive integers k, m, n,
$$(m \mid k \ \wedge \ n \mid k) \Longrightarrow (m \cdot n) \mid k$$

6. Prove that for all integers d, k, l, m, n,
   a) $d \mid m \wedge d \mid n \Longrightarrow d \mid (m+n)$
   b) $d \mid m \Longrightarrow d \mid k \cdot m$
   c) $d \mid m \wedge d \mid n \Longrightarrow d \mid (k \cdot m + l \cdot n)$
7. Prove that for all integers n,
$$30 \mid n \iff (2 \mid n \;\wedge\; 3 \mid n \;\wedge\; 5 \mid n)$$
8. Show that for all integers m and n,
$$(m \mid n \wedge n \mid m) \Longrightarrow (m = n \vee m = -n)$$
9. Prove or disprove that: For all positive integers k, m, n,
$$k \mid (m \cdot n) \Longrightarrow k \mid m \vee k \mid n$$
10. Let $P(m)$ be a statement for m ranging over the natural numbers, and consider the following derived statement (with n also ranging over the natural numbers):
$$P^{\#}(n) \triangleq \forall k \in \mathbb{N}.\ 0 \leq k \leq n \Longrightarrow P(k)$$
   a) Show that, for all natural numbers ℓ, $P^{\#}(\ell) \Longrightarrow P(\ell)$.
   b) Exhibit a concrete statement $P(m)$ and a specific natural number n for which the following statement *does not* hold:
$$P(n) \Longrightarrow P^{\#}(n)$$
   c) Prove the following:
      - $P^{\#}(0) \iff P(0)$
      - $\forall n \in \mathbb{N}.\ \big(P^{\#}(n) \Longrightarrow P^{\#}(n+1)\big) \iff \big(P^{\#}(n) \Longrightarrow P(n+1)\big)$
      - $\big(\forall m \in \mathbb{N}.\ P^{\#}(m)\big) \iff (\forall m \in \mathbb{N}.\ P(m))$

1.3. Optional exercises

1. A series of questions about the properties and relationship of triangular and square numbers (adapted from David Burton).
   a) A natural number is said to be *triangular* if it is of the form $\sum_{i=0}^{k} i = 0 + 1 + \cdots + k$, for some natural k. For example, the first three triangular numbers are $t_0 = 0$, $t_1 = 1$ and $t_2 = 3$.

      Find the next tree triangular numbers t_3, t_4 and t_5.
   b) Find a formula for the k^{th} triangular number t_k.
   c) A natural number is said to be *square* if it is of the form k^2 for some natural number k.

Show that n is triangular iff $8 \cdot n + 1$ is a square. (Plutarch, circ. 100BC)

d) Show that the sum of every two consecutive triangular numbers is square. (Nicomachus, circ. 100BC)

e) Show that, for all natural numbers n, if n is triangular, then so are $9 \cdot n + 1$, $25 \cdot n + 3$, $49 \cdot n + 6$ and $81 \cdot n + 10$. (Euler, 1775)

f) Prove the generalisation: For all n and k natural numbers, there exists a natural number q such that $(2n+1)^2 \cdot t_k + t_n = t_q$. (Jordan, 1991, attributed to Euler)

2. Let $P(x)$ be a predicate on a variable x and let Q be a statement not mentioning x. Show that the following equivalence holds:

$$\Big(\big(\exists x.\, P(x)\big) \Longrightarrow Q\Big) \Longleftrightarrow \Big(\forall x.\, \big(P(x) \Longrightarrow Q\big)\Big)$$