

Foundation axiom

The membership relation is well-founded.

Thereby, providing a

Principle of \in -Induction .

Well-founded Relations

Definition: A relation on a set $< \subseteq A \times A$ is well-founded if every non-empty subset $S \subseteq A$ has an element $m \in S$ that is minimal.

$$\hookrightarrow \forall x \in A. x < m \Rightarrow x \notin S$$

equivalently

$$\neg (\exists x \in S. x > m)$$

Proposition: A relation $< \subseteq A \times A$ is well-founded \uparrow , and only if, there are no infinite sequences

$a_0, a_1, \dots, a_i, \dots$

that are descending in the sense that

$a_0 \succ a_1 \succ \dots \succ a_i \succ \dots$

Principle of Well-Founded Induction

Let $< \subseteq A \times A$ be a well-founded relation and let $P(a)$ be a statement for $a \in A$.

To prove $\forall a \in A. P(a)$

Show that, for $x \in A$, the induction

hypothesis

implies

$$\forall y \in A. y < x \Rightarrow P(y) \quad (\text{IH})$$

$$P(x)$$

Principle of Well-Founded Induction

For a well-founded relation $< \subseteq A \times A$,

$$\left(\forall x \in A. \left[\forall y \in A. y < x \Rightarrow P(y) \right] \Rightarrow P(x) \right)$$

$$\Rightarrow \forall a \in A. P(a)$$

Example: The strictly less than relation on natural numbers is well-founded.

Principle of Well-Founded Induction

$$\left(\forall x \in \mathbb{N}. \left[\forall y \in \mathbb{N}. y < x \Rightarrow P(y) \right] \Rightarrow P(x) \right)$$

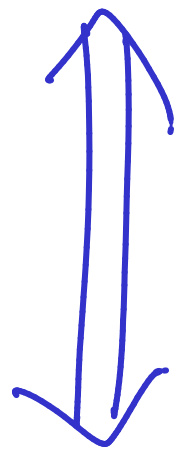
$$\Rightarrow \forall a \in \mathbb{N}. P(a)$$

To prove $\forall a \in \mathbb{N}. P(a)$

show

$$\left(\forall x \in \mathbb{N}. \left[\forall y \in \mathbb{N}. y < x \Rightarrow P(y) \right] \Rightarrow P(x) \right)$$

$$\forall x \in \mathbb{N}. \underbrace{Q(x)}$$



$$\stackrel{\text{ll def}}{=} [\forall y \in \mathbb{N}. y < x \Rightarrow P(y)] \Rightarrow P(x)$$

$$Q(0) \wedge \forall n \in \mathbb{N}. Q(n+1)$$

$$Q(0) \equiv (\forall y \in \mathbb{N}. y < 0 \Rightarrow P(y)) \Rightarrow P(0)$$

$$\Leftrightarrow P(0)$$

$$\forall n \in \mathbb{N}. Q(n+1)$$

$$\equiv \forall n \in \mathbb{N}. [\forall y \in \mathbb{N}. y < n+1 \Rightarrow P(y)] \Rightarrow P(n+1)$$

$$\Leftrightarrow \forall n \in \mathbb{N}. (\forall y \in \mathbb{N}. y \leq n \Rightarrow P(y)) \Rightarrow P(n+1)$$

$$\forall n \in \mathbb{N}. Q(n)$$

$$\Leftrightarrow P(0) \wedge (\forall n \in \mathbb{N}. (\forall y \in \mathbb{N}. y \leq n \Rightarrow P(y)) \Rightarrow P(n+1))$$

Exercise: Spell out the principle of well-founded induction for $\mathcal{N} \times \mathcal{N}$ ordered by

$$(1) (i, j) < (k, l) \stackrel{\text{def}}{\iff} (i < k) \wedge (j < l)$$

$$(2) (i, j) < (k, l) \stackrel{\text{def}}{\iff} (i < k) \vee [(i = k) \wedge (j < l)]$$