Foundation axiom

$\boxed{\text{The membership relation is well-founded.}}$

Thereby, providing a

Principle of $\in$-Induction .

Well-founded Relations

Definition: A relation on a set $< \subseteq A \times A$ is well-founded if every non-empty subset $S \subseteq A$ has an element $m \in S$ that is minimal.

$$\forall x \in A.\ x < m \Rightarrow x \notin S$$

equivalently

$$\neg(\exists x \in S.\ x < m)$$

Proposition: A relation $\prec \subseteq A \times A$ is well-founded if, and only if, there are no infinite sequences

$$a_0, a_1, \cdots, a_i, \cdots$$

that are descending in the sense that

$$a_0 \succ a_1 \succ \cdots \succ a_i \succ \cdots$$

Principle of Well-Founded Induction

Let $\prec \subseteq A \times A$ be a well-founded relation and let $P(a)$ be a statement for $a \in A$.

To prove

$$\forall a \in A.\ P(a)$$

Show that, for $x \in A$, the induction hypothesis

$$\forall y \in A.\ y \prec x \Rightarrow P(y) \qquad (IH)$$

implies

$$P(x)$$

Principle of Well-Founded Induction

For a well-founded relation $< \subseteq A \times A$,

$$\left(\forall x \in A.\ \left[\forall y \in A.\ y < x \Rightarrow P(y) \right] \Rightarrow P(x) \right)$$

$$\Rightarrow \forall a \in A.\ P(a)$$

Example: The strictly less than relation on natural numbers is well-founded.

Principle of Well-Founded Induction

$$\left(\forall x \in \mathbb{N}.\ \left[\forall y \in \mathbb{N}.\ y < x \Rightarrow P(y)\right] \Rightarrow P(x)\right)$$

$$\Rightarrow \forall a \in \mathbb{N}.\ P(a)$$

To prove $\forall a \in \mathbb{N}.\ P(a)$

show

$$\left(\forall x \in \mathbb{N}.\ \left[\forall y \in \mathbb{N}.\ y < x \Rightarrow P(y)\right] \Rightarrow P(x)\right)$$

$$\forall x \in \mathbb{N}.\ \underbrace{Q(x)}_{\text{def}}$$

$$\left[\forall y \in \mathbb{N}.\ y < x \Rightarrow P(y)\right] \Rightarrow P(x)$$

$$\Updownarrow$$

$$Q(0) \wedge \forall n \in \mathbb{N}.\ Q(n+1)$$

$$Q(0) \equiv \left(\forall y \in \mathbb{N}.\ y < 0 \Rightarrow P(y)\right) \Rightarrow P(0)$$

$$\Leftrightarrow P(0)$$

$$\forall n \in \mathbb{N}.\ Q(n+1)$$

$$\equiv \forall n \in \mathbb{N}.\ [\forall y \in \mathbb{N}.\ y < n+1 \Rightarrow P(y)] \Rightarrow P(n+1)$$

$$\Leftrightarrow \forall n \in \mathbb{N}.\ (\forall y \in \mathbb{N}.\ y \leq n \Rightarrow P(y)) \Rightarrow P(n+1)$$

$$\forall n \in \mathbb{N}.\ Q(n)$$

$$\Leftrightarrow P(0) \wedge (\forall n \in \mathbb{N}.\ (\forall y \in \mathbb{N}.\ y \leq n \Rightarrow P(y)) \Rightarrow P(n+1))$$

Exercise: Spell out the principle of well-founded induction for $\mathbb{N} \times \mathbb{N}$ ordered by

(1) $(i,j) < (k,l) \overset{\text{def}}{\Longleftrightarrow} (i<k) \wedge (j<l)$

(2) $(i,j) < (k,l) \overset{\text{def}}{\Longleftrightarrow} (i<k) \vee \left[(i=k) \wedge (j<l) \right]$