Replacement axiom

The direct image of every definable functional property on a set is a set.

From a mapping

$$i \longmapsto f(i)$$

The replacement axiom allows the construction of a set

$$\{ f(i) \mid i \in I \}$$

for i ranging over an indexing set I.

Given a mapping

$$i \longmapsto A_i$$

with each A_i a set, for i ranging over an indexing set I, we may construct the set

$$\{A_i \mid i \in I\}$$

and thereby the set

$$\bigcup \{A_i \mid i \in I\}.$$

Indexed Disjoint Unions

$i \longmapsto A_i$ a set

$i \longmapsto \{i\} \times A_i$ a set

$\bigcup \{ \{i\} \times A_i \mid i \in I \}$ I a set

$\|$

$\biguplus_{i \in I} A_i$

Examples

(1) $n \longmapsto A^n \; (= \underbrace{A \times \cdots \times A}_{n \text{ times}})$

$$\biguplus_{n \in \mathbb{N}} A^n = A^*$$

The set of finite sequences (or lists) on A

(2) For sets A and B,

$$\mathcal{P}_{fin}(A) \ni S \longmapsto (S \Rightarrow B)$$

$$\mathcal{P}_{fin}(A) \overset{def}{=} \{S \subseteq A \mid S \text{ finite}\}$$

$$\biguplus_{S \in \mathcal{P}_{fin}(A)} (S \Rightarrow B) = (A \Rightarrow_{fin} B)$$

The set of all partial functions from A to B with finite domain of definition

Indexed Intersections

$i \longmapsto A_i$ a set

$$\left\{ x \in \bigcup_{i \in I} A_i \;\middle|\; \forall i \in I .\, x \in A_i \right\}$$

$// \underline{\text{def}}$

$$\bigcap_{i \in I} A_i$$

Indexed Products

$i \longmapsto A_i$ a set

I a set

$$\left\{ \alpha : I \Longrightarrow \bigcup_{i \in I} A_i \;\middle|\; \forall i \in I.\ \alpha(i) \in A_i \right\}$$

// def

$$\prod_{i \in I} A_i$$

Example: $n \in \mathbb{N}$

$$\prod_{i \in [n]} A_i \cong (A_0 \times A_1 \times \cdots \times A_{n-1})$$

Set-indexed constructions

For every mapping associating a set A_i to each element of a set I, we have the set

$$\bigcup_{i \in I} A_i = \bigcup \{ A_i \mid i \in I \} = \{ a \mid \exists i \in I.\, a \in A_i \} \quad .$$

Examples:

1. Indexed disjoint unions:

$$\biguplus_{i \in I} A_i = \bigcup_{i \in I} \{i\} \times A_i$$

2. Finite sequences on a set A:

$$A^* = \biguplus_{n \in \mathbb{N}} A^n$$

3. Finite partial functions from a set A to a set B:

$$(A \rightharpoonup_{\text{fin}} B) = \biguplus_{S \in \mathcal{P}_{\text{fin}}(A)} (S \Rightarrow B)$$

where

$$\mathcal{P}_{\text{fin}}(A) = \{ S \subseteq A \mid S \text{ is finite} \}$$

4. Non-empty indexed intersections: for $I \neq \emptyset$,

$$\bigcap_{i \in I} A_i = \{ x \in \bigcup_{i \in I} A_i \mid \forall i \in I .\, x \in A_i \}$$

5. Indexed products:

$$\prod_{i \in I} A_i = \left\{ \alpha \in (I \Rightarrow \bigcup_{i \in I} A_i) \;\middle|\; \forall i \in I .\, \alpha(i) \in A_i \right\}$$

Proposition 153 *An enumerable indexed disjoint union of enumerable sets is enumerable.*

PROOF: Let $e : \mathbb{N} \twoheadrightarrow I$ be an enumeration and, for every $i \in I$, let $\alpha_i : \mathbb{N} \twoheadrightarrow A_i$ be an enumeration.

Then, the function

$$\begin{aligned} \mathbb{N} \times \mathbb{N} &\longrightarrow \biguplus_{i \in I} A_i \\ (m, n) &\longmapsto (e(m), \alpha_{e(m)}(n)) \end{aligned}$$

is surjective, and one may construct an enumeration

$$\mathbb{N} \xrightarrow{\cong} \mathbb{N} \times \mathbb{N} \twoheadrightarrow \biguplus_{i \in I} A_i .$$

∎

Corollary 155 *If* X *and* A *are countable sets then so are* A^*, $\mathcal{P}_{\mathrm{fin}}(A)$, *and* $(X \rightharpoonup_{\mathrm{fin}} A)$.

There are non-computable infinite sequences of bits.