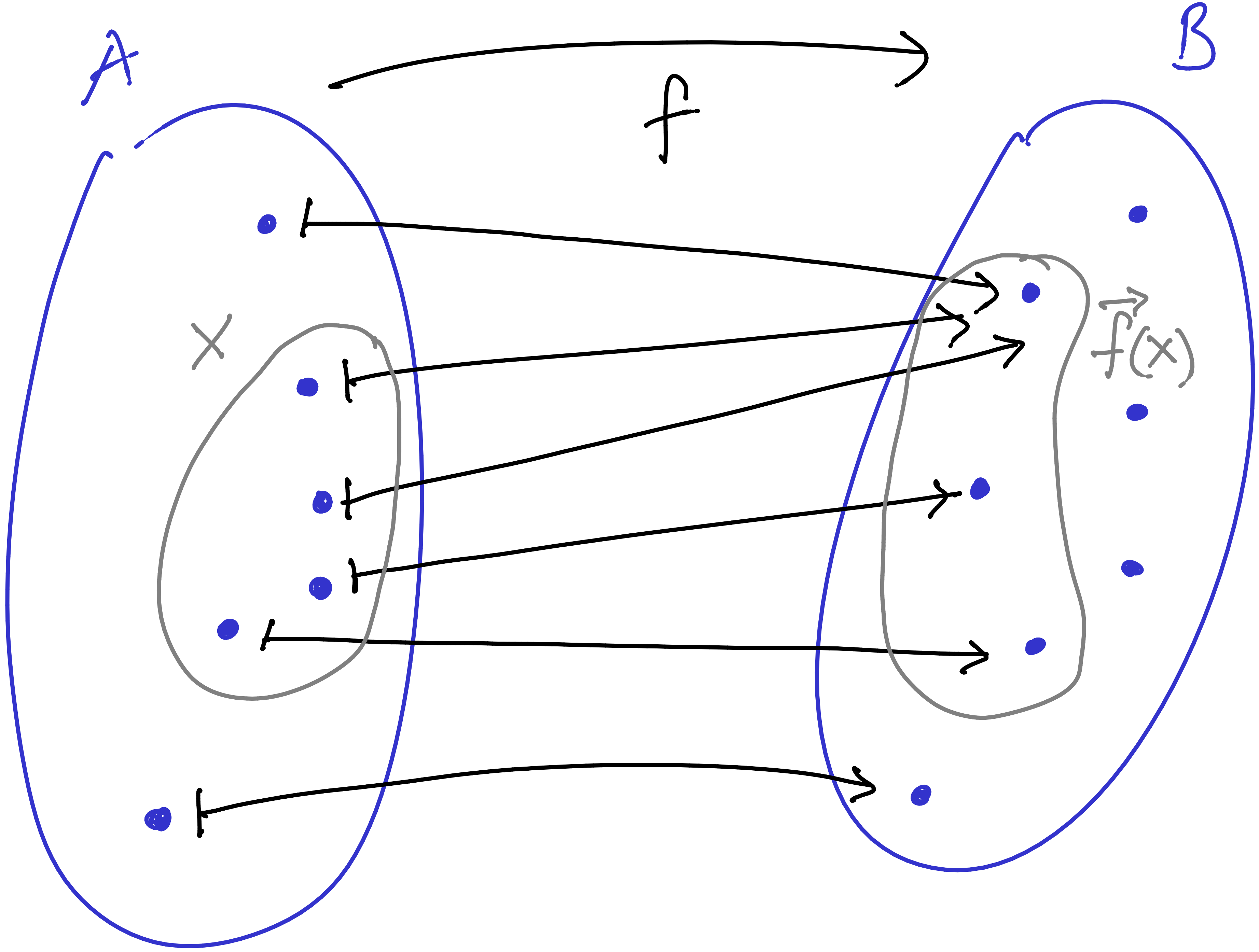
Direct and Inverse Images

Functional Images

Definition Let $f: A \to B$ be a function. The <u>direct image</u> of $X \subseteq A$ under f is the set $\vec{f}(X) \subseteq B$, defined as

$$\vec{f}(X) = \{ b \in B \mid \exists x \in X.\ f(x) = b \}$$

$$= \{ f(x) \in B \mid x \in X \}$$



A
f
B
X
$\vec{f}(X)$

Proposition For all functions $f : A \to B$, the mapping

$$A \ni a \longmapsto f(a)$$

determines a function

$$f' : A \longrightarrow \vec{f}(A)$$

that is surjective.
Moreover, whenever $f : A \to B$ is injective, $f' : A \to \vec{f}(A)$ is bijective.

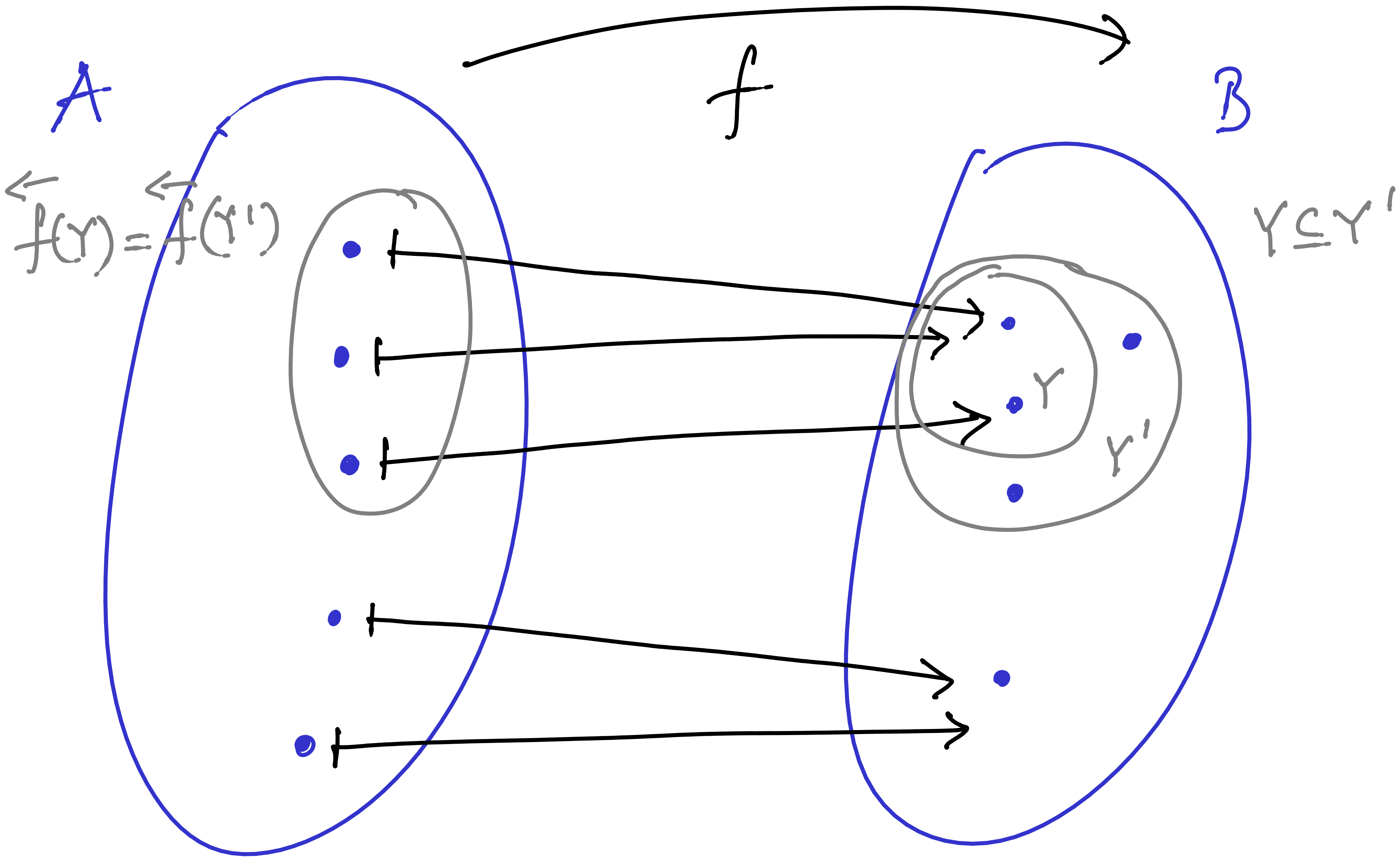
Injective functions preserve cardinality

Corollary For an injective function $f : A \rightarrowtail B$,

$$\forall X \subseteq A.\ X \cong \vec{f}(X).$$

Definition: Let $f : A \to B$ be a function. The <u>inverse image</u> of $Y \subseteq B$ is the set $\overleftarrow{f}(Y) \subseteq A$ defined as

$$\overleftarrow{f}(Y) = \{ a \in A \mid f(a) \in Y \}.$$



f
A
B
$\overleftarrow{f}(Y) = \overleftarrow{f}(Y')$
$Y \subseteq Y'$
Y
Y'

Proposition: For $f: A \to B$, the mapping

$$B \supseteq Y \longmapsto \overleftarrow{f}(Y) \subseteq A$$

determines a function

$$\mathcal{P}(B) \longrightarrow \mathcal{P}(A)$$

that preserves the Boolean algebra structure of powersets.

E.g. $\overleftarrow{f}(Y^c) = \{ a \in A \mid f(a) \in Y^c \}$

$= \{ a \in A \mid f(a) \notin Y \}$

$= \{ a \in A \mid f(a) \in Y \}^c$

$= \left(\overleftarrow{f}(Y)\right)^c$

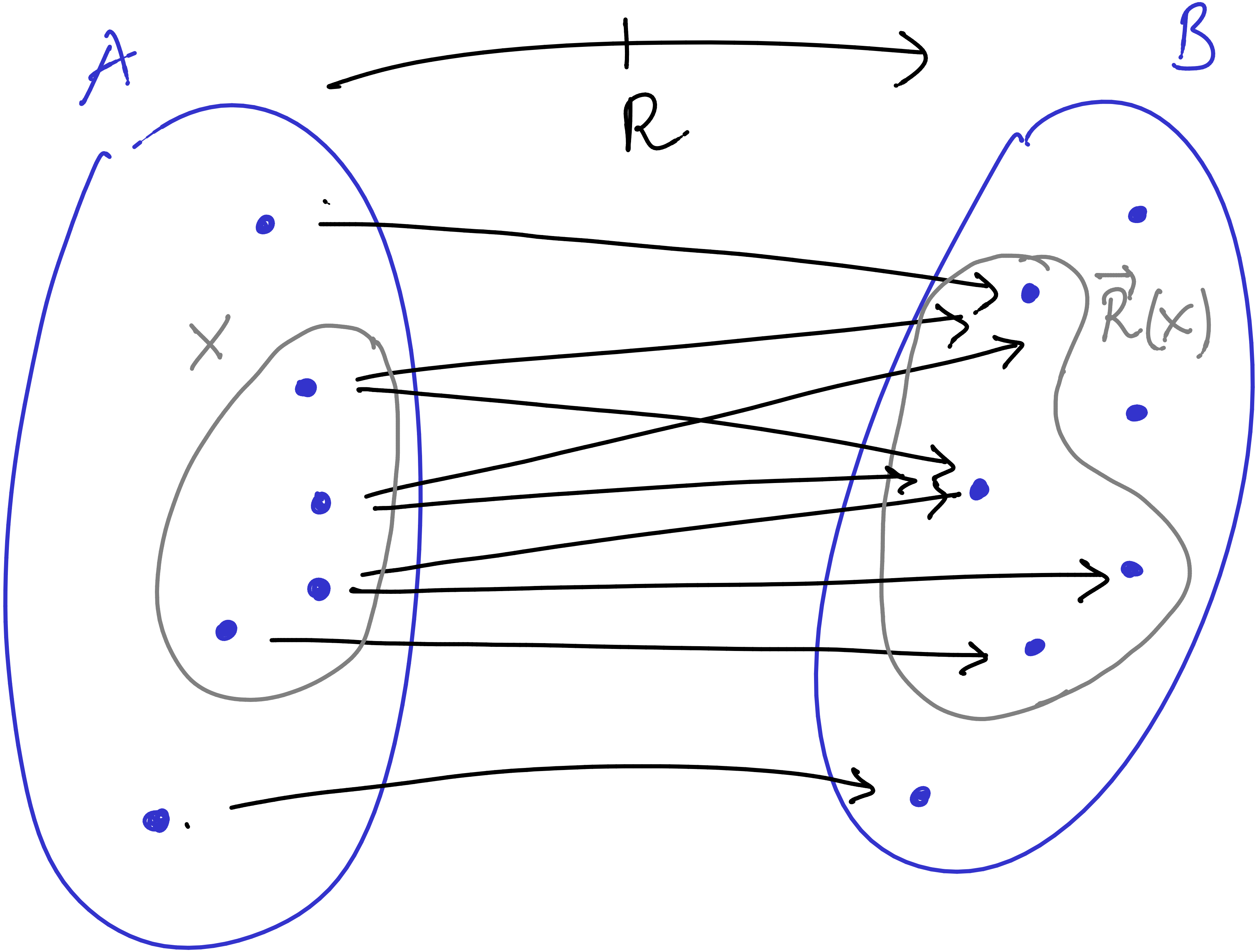
Relational images

Definition 150 *Let* $R : A \longrightarrow\!\!\!\!\!\!\!+\;\; B$ *be a relation.*

- *The* direct image *of* $X \subseteq A$ *under* R *is the set* $\overrightarrow{R}(X) \subseteq B$, *defined as*

$$\overrightarrow{R}(X) \;=\; \{\, b \in B \mid \exists x \in X.\, x \mathrel{R} b \,\} \;.$$

NB *This construction yields a function* $\overrightarrow{R} : \mathcal{P}(A) \to \mathcal{P}(B)$.



A
B
R
X
$\vec{R}(X)$

Relational Inverse Images

- For $R : A \nrightarrow B$, $x \in A$ is to be thought of an input mapping to the set of output values $\overrightarrow{R}\{x\} = \{y \in B \mid xRy\}$.
- For $Y \subseteq B$, thought of as describing a property for outputs, $\overleftarrow{R}(Y)$

$$\overleftarrow{R}(Y) \overset{\text{def}}{=} \{x \in X \mid \overrightarrow{R}\{x\} \subseteq Y\}$$

- *The* inverse image *of* $Y \subseteq B$ *under* R *is the set* $\overleftarrow{R}(Y) \subseteq A$, *defined as*

$$\overleftarrow{R}(Y) \ = \ \{ a \in A \mid \forall b \in B.\, a\,R\,b \implies b \in Y \}$$

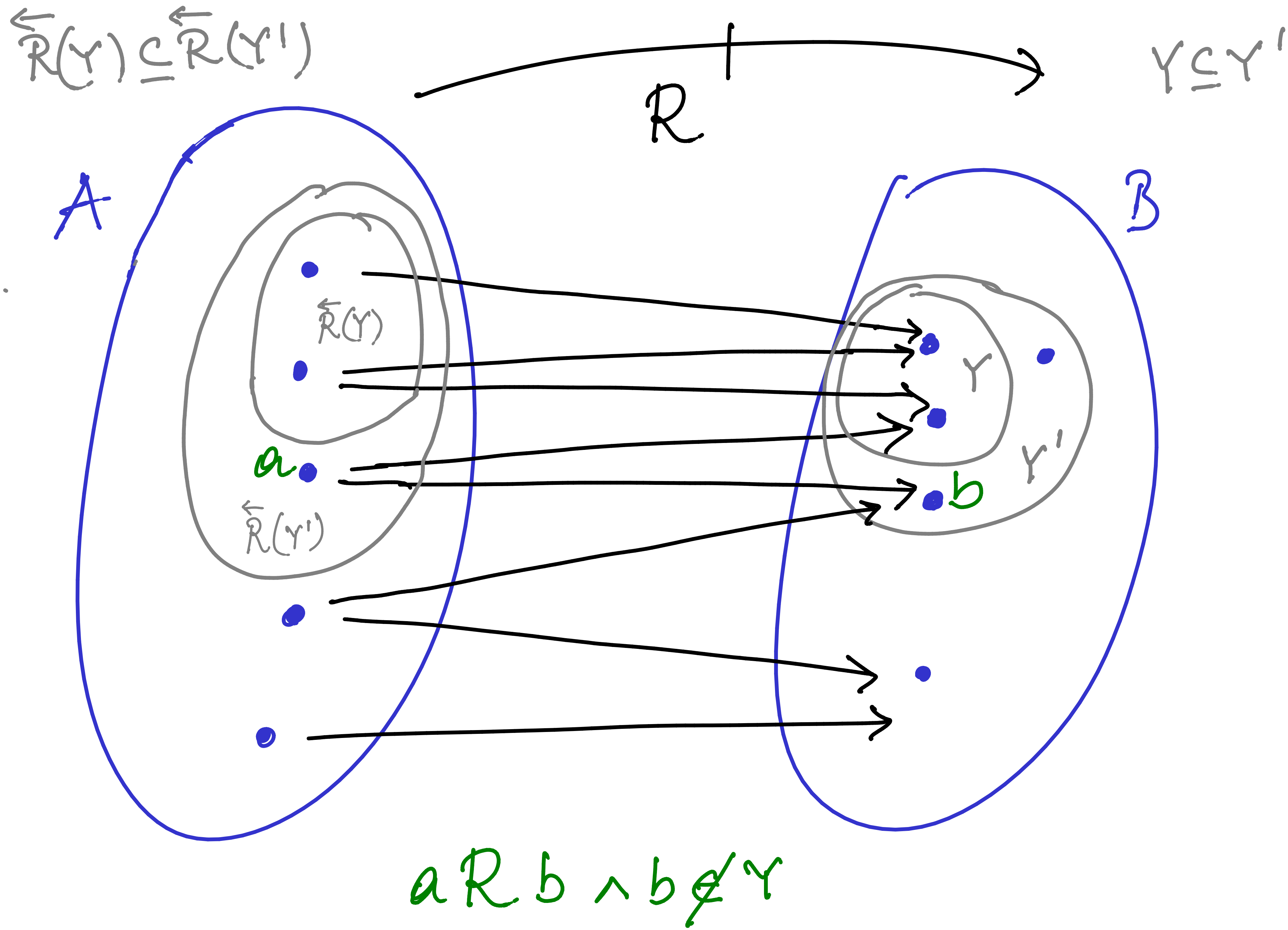
NB: $\forall a \in A.$

$$a \notin \overleftarrow{R}(Y)$$

$$\iff$$

$$\exists b \in B.\ a\,R\,b \wedge b \notin Y$$

NB *This construction yields a function* $\overleftarrow{R} : \mathcal{P}(B) \to \mathcal{P}(A)$.



$\overleftarrow{R}(Y) \subseteq \overleftarrow{R}(Y')$
R
$Y \subseteq Y'$
A
B
$\overleftarrow{R}(Y)$
$\overleftarrow{R}(Y')$
a
b
Y
Y'
$aRb \wedge b \notin Y$