NB: Every section-retraction pair

\[ A \xrightarrow{r} B \xleftarrow{s} B \]

is such that

- the retraction \( r : A \to B \) is a surjection
- the section \( s : B \to A \) is an injection
Injections

Definition 145  A function \( f : A \to B \) is said to be \underline{injective}, or an \underline{injection}, and indicated \( f : A \hookrightarrow B \) whenever

\[
\forall a_1, a_2 \in A. (f(a_1) = f(a_2)) \implies a_1 = a_2.
\]

\( f : A \to B \) fails to be injective if there are \( a_1, a_2 \in A \) such that

\( a_1 \neq a_2 \) but \( f(a_1) = f(a_2) \).
Proposition: Every section is an injection.

Proof: Let \( s : A \rightarrow B \) be a section; that is, such that there is \( r : B \rightarrow A \) with the property \( ros = 1_A \).

RTP: For \( a, a' \in A \), \( s(a) = s(a') \Rightarrow a = a' \).

Let \( a, a' \in A \) and assume \( s(a) = s(a') \).

Then, \( a = r(s(a)) = r(s(a')) = a' \). \( \Box \)
Proposition: Let $X$ be a set.

(i) The unique function from the empty set $\emptyset$ to $X$ is an injection, and

(ii) it is a section if, and only if, $X = \emptyset$.

Proof: The unique function $\emptyset \to X$ is given by the empty relation and so it is vacuously an injection. Moreover, there is only one relation from $X \to \emptyset$, namely the empty relation and this is a...
function \( f \) if \( X = \emptyset \); in which case the function \( \phi \mapsto \phi \) is the identity and so trivially a section.
Theorem 146  The identity function is an injection, and the composition of injections yields an injection.

The set of injections from $A$ to $B$ is denoted

$$\text{Inj}(A, B)$$

and we thus have

$$\text{Sur}(A, B) \subseteq \text{Bij}(A, B) \subseteq \text{Inj}(A, B) \subseteq \text{Fun}(A, B) \subseteq \text{PFun}(A, B) \subseteq \text{Rel}(A, B)$$

with

$$\text{Bij}(A, B) = \text{Sur}(A, B) \cap \text{Inj}(A, B)$$.
Proposition 147  For all finite sets $A$ and $B$,

$$\#\text{Inj}(A, B) = \begin{cases} 
(\#_B) \cdot (\#_A)! & \text{, if } \#_A \leq \#_B \\
0 & \text{, otherwise}
\end{cases}$$

**Proof Idea:**

$A = \{a_1, a_2, \ldots, a_m\}$

$B = \{b_1, b_2, \ldots, b_n\}$

(1) $n < m$

$$\begin{array}{cccccc}
a_1 & a_2 & \ldots & a_{n-1} & a_n & a_{n+1} & \ldots & a_m \\
| & | & \searrow & \downarrow & \nearrow & \searrow & \ldots & \nearrow \\
b_1 & b_2 & \ldots & b_{n-1} & b_n
\end{array}$$
(2) $n \geq m$

\[ a_1 \mapsto b_{j_1}, \quad n \text{ choices} \]
\[ a_2 \mapsto b_{j_2}, \quad (n-1) \text{ choices} \]
\[ \vdots \]
\[ a_m \mapsto b_{j_m}, \quad (n-(m-1)) \text{ choices} \]

\[ \# \text{Inj}^m(A,B) = n \times (n-1) \times \cdots \times (n-m+1) \]

\[ = \binom{n}{m} \times m! \]