NB: Every section-retraction pair

$$\begin{tikzcd} A \arrow[r, shift left, "r"] & B \arrow[l, bend left, "s"] \end{tikzcd} \qquad r \circ s = \mathrm{id}_B$$

is such that

- the retraction $r : A \to B$ is a surjection

and

- the section $s : B \to A$ is an injection

Injections

Definition 145 *A function* $f : A \to B$ *is said to be* injective, *or an* injection, *and indicated* $f : A \rightarrowtail B$ *whenever*

$$\forall a_1, a_2 \in A. \left(f(a_1) = f(a_2)\right) \implies a_1 = a_2 \quad .$$

$f : A \to B$ fails to be injective if there are $a_1, a_2 \in A$ such that $a_1 \neq a_2$ but $f(a_1) = f(a_2)$.

Proposition: Every section is an injection.

PROOF: Let $s: A \to B$ be a section; That is, such That There is $r: B \to A$ with the property $r \circ s = \mathrm{id}_A$.

<u>RTP</u>: For $a, a' \in A$. $s(a) = s(a') \Rightarrow a = a'$.

Let $a, a' \in A$ and assume $s(a) = s(a')$.

Then, $a = r(s(a)) = r(s(a')) = a'$. $\boxtimes$

Proposition: Let X be a set.

(i) The unique function from the empty set $\emptyset$ to X is an injection, and

(ii) it is a section if, and only if, $X=\emptyset$.

PROOF: The unique function $\emptyset \to X$ is given by the empty relation and so it is vacuously an injection. Moreover, there is only one relation from $X \nrightarrow \emptyset$, namely the empty relation and this is a function iff $X = \emptyset$; in which case the function $\emptyset \to \emptyset$ is the identity and so trivially a section. $\boxtimes$

Theorem 146 *The identity function is an injection, and the composition of injections yields an injection.*

The set of injections from A to B is denoted

$$\mathrm{Inj}(A, B)$$

and we thus have

$$\begin{tikzcd}[column sep=small, row sep=small]
 & \mathrm{Sur}(A,B) \arrow[dr, phantom, "\subseteq" sloped] & \\
\mathrm{Bij}(A,B) \arrow[ur, phantom, "\subseteq" sloped] \arrow[dr, phantom, "\subseteq" sloped] & & \mathrm{Fun}(A,B) \subseteq \mathrm{PFun}(A,B) \subseteq \mathrm{Rel}(A,B) \\
 & \mathrm{Inj}(A,B) \arrow[ur, phantom, "\subseteq" sloped] &
\end{tikzcd}$$

with

$$\mathrm{Bij}(A, B) = \mathrm{Sur}(A, B) \cap \mathrm{Inj}(A, B) \quad .$$

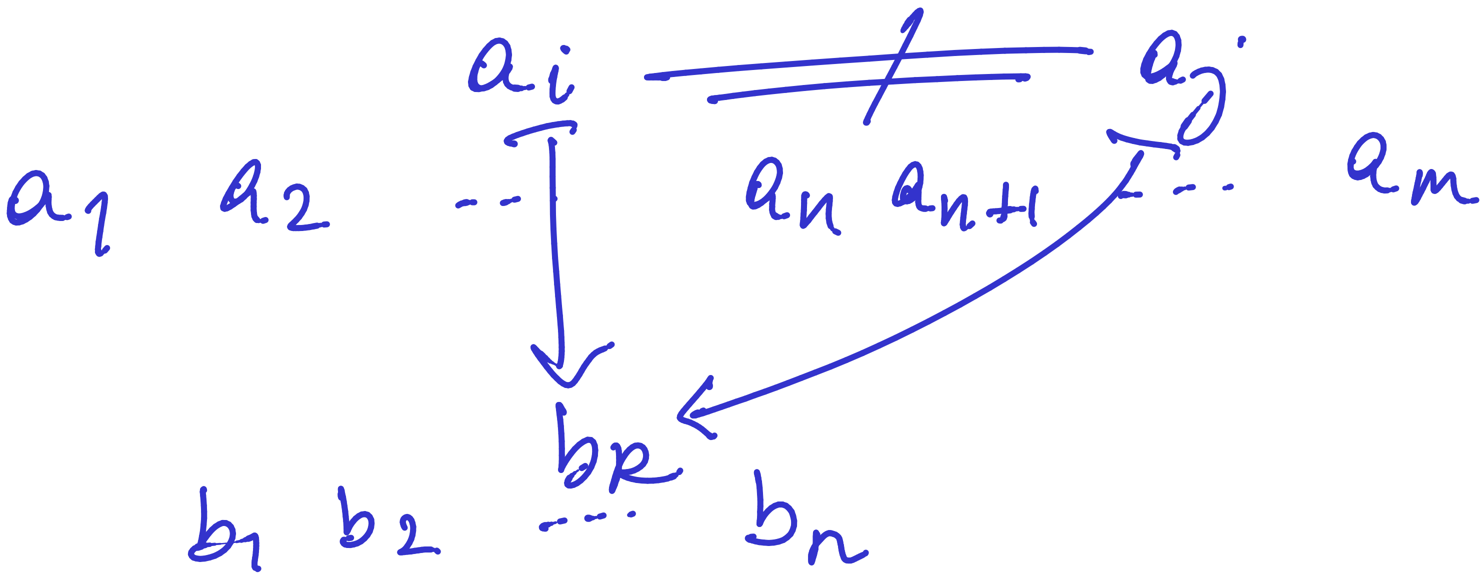
Proposition 147 *For all finite sets* A *and* B*,*

$$\#\mathrm{Inj}(A,B) = \begin{cases} \binom{\#B}{\#A} \cdot (\#A)! & \text{, if } \#A \le \#B \\ 0 & \text{, otherwise} \end{cases}$$

PROOF IDEA: $A = \{a_1, a_2, \ldots, a_m\}$

$B = \{b_1, b_2, \ldots, b_n\}$

(1) $n < m$



(2) $n \geq m$

$$a_1 \longmapsto b_{j_1} \quad n \text{ choices}$$

$$a_2 \longmapsto b_{j_2} \quad (n-1) \text{ choices}$$

$$\vdots$$

$$a_m \longmapsto b_{j_m} \quad (n-(m-1)) \text{ choices}$$

$$\# \underline{\mathrm{Inj}}(A, B) = n \times (n-1) \times \cdots \times (n-m+1)$$

$$= \binom{n}{m} \times m!$$

$\boxtimes$