Uncountable Cardinality

Theorem The sets

$$\mathcal{P}(\mathbb{N}) \cong (\mathbb{N} \Rightarrow [2]) \cong (\mathbb{N} \Rightarrow [1])$$

are not countable.

PROOF: By way of contradiction, suppose that there is an enumeration

$$e : \mathbb{N} \twoheadrightarrow (\mathbb{N} \Rightarrow [2])$$

For $n \in \mathbb{N}$, each

$$e(n) : \mathbb{N} \to [2]$$

may be visualised as an infinite sequence of 0's and 1's

$$e(n) = \underbrace{e(n)(0), e(n)(1), \ldots, e(n)(i), \ldots}_{i \in \mathbb{N}}$$

We will derive a contradiction by showing that there is an infinite sequence of

0's and 1's

$s = s(0), s(1), \cdots, s(i), \cdots$
$i \in \mathbb{N}$

That is different from each $e(n)$ for all $n \in \mathbb{N}$.

To guarantee $s \neq e(0)$ we will define

$s(0) = \overline{e(0)(0)}$

To guarantee $s \neq e(1)$ we will define

$s(1) = \overline{e(1)(1)}$

To guarantee that s is different from $e(n)$ we define

$$s(n) = \overline{e(n)(n)} \quad \text{for all } n \in \mathbb{N}.$$

Indeed, as e is surjective, there exists $i \in \mathbb{N}$ such that $e(i) = s$. But then

$$e(i)(i) = s(i) = \overline{e(i)(i)}$$

A contradiction. $\boxtimes$

Analogously, suppose there is an enumeration

$$e : \mathbb{N} \twoheadrightarrow \mathcal{P}(\mathbb{N})$$

Define

$$S \in \mathcal{P}(\mathbb{N})$$
$$\parallel$$
$$\{ n \in \mathbb{N} \mid n \notin e(n) \}$$

Then, there exists $i \in \mathbb{N}$ such that

$$e(i) = S$$

It follows

$$i \in e(i) \Leftrightarrow i \in S \Leftrightarrow i \notin e(i) \quad ↯ \qquad \boxtimes$$

Corollary: The sets

$$[0,1] \cong \mathbb{R}$$

are uncountable

PROOF: Note that

$$(\mathbb{N} \Rightarrow [2]) \xrightarrow{\cong} [0,1]$$

$$s \longmapsto \sum_{n \in \mathbb{N}} \frac{s_n}{2^{n+1}}$$

is a bijection. $\boxtimes$

Unbounded cardinality

Theorem 156 (Cantor's diagonalisation argument) *For every set A, no surjection from A to $\mathcal{P}(A)$ exists.*

PROOF: By contradiction assume a surjection

$$e : A \twoheadrightarrow \mathcal{P}(A)$$

Define

$$S \in \mathcal{P}(A)$$

$$\stackrel{\text{def}}{=} \{ a \in A \mid a \notin e(a) \}$$

Then there exists $i \in A$ such that $e(i) = S$.

It follows: $i \in e(i) \Leftrightarrow i \in S \Leftrightarrow i \notin e(i)$. $\lightning$ $\boxtimes$

Corollary For all sets A, there is no surjection from A to $(A \to [2])$.

Definition 157 *A* fixed-point *of a function* $f : X \to X$ *is an element* $x \in X$ *such that* $f(x) = x$.

Theorem 158 (Lawvere's fixed-point argument) *For sets* A *and* X*, if there exists a surjection* $A \twoheadrightarrow (A \Rightarrow X)$ *then every function* $X \to X$ *has a fixed-point; and hence* X *is a singleton.*

PROOF: For sets A and X, and let

$$e : A \twoheadrightarrow (A \Rightarrow X)$$

RTP: For every function $f : X \to X$ there is $x \in X$ such that $f(x) = x$

Let $f : X \to X$.

Consider

$$s: A \to X$$
$$a \mapsto f(e(a)(a))$$

There exists $i \in A$ such that

$$e(i) = s$$

Then

$$e(i)(i) = s(i) = f(e(i)(i))$$

Hence $e(i)(i) \in X$ is a fixed point of f. $\boxtimes$