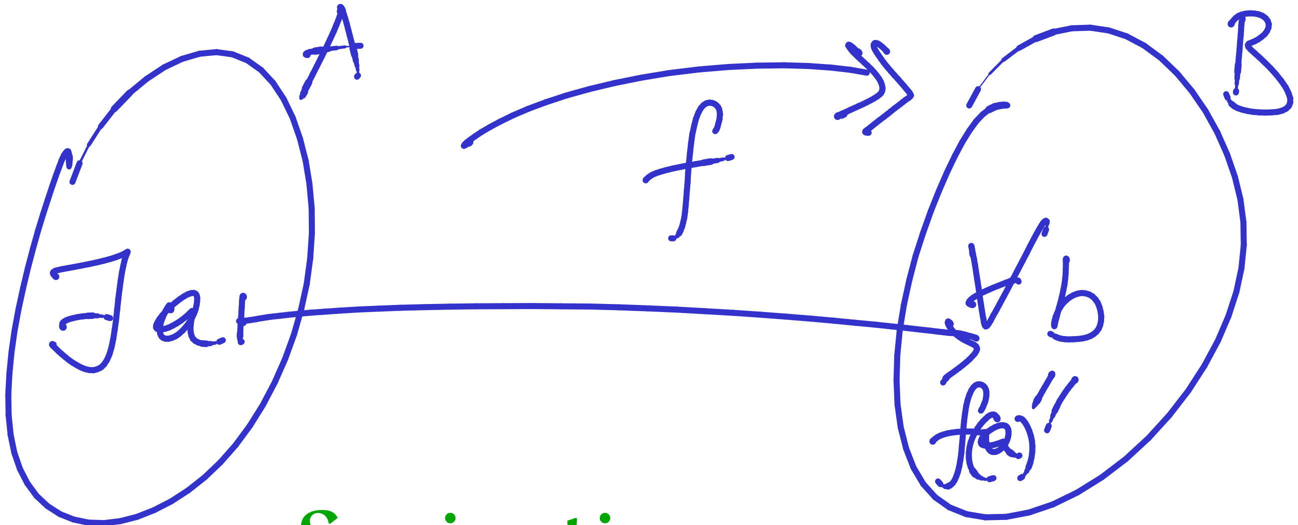
Bijections

Proposition 138 *For a function* $f : A \to B$*, the following are equivalent.*

1. f *is bijective.*
2. $\forall\, b \in B.\, \exists!\, a \in A.\, f(a) = b$.
3. $$\begin{array}{l} \big(\forall\, b \in B.\, \exists\, a \in A.\, f(a) = b\big) \quad \text{surjections} \\ \land \\ \big(\forall\, a_1, a_2 \in A.\, f(a_1) = f(a_2) \implies a_1 = a_2\big) \quad \text{injections} \end{array}$$



Surjections

Definition 139 *A function* $f : A \to B$ *is said to be* surjective*, or a* surjection*, and indicated* $f : A \twoheadrightarrow B$ *whenever*

$$\forall\, b \in B.\, \exists\, a \in A.\, f(a) = b \quad .$$

NB: $\{ f(a) \mid a \in A \} = B$.

Theorem 140 *The identity function is a surjection, and the composition of surjections yields a surjection.*

The set of surjections from A to B is denoted

$$\mathrm{Sur}(A, \mathrm{B})$$

and we thus have

$$\mathrm{Bij}(A, \mathrm{B}) \subseteq \mathrm{Sur}(A, \mathrm{B}) \subseteq \mathrm{Fun}(A, \mathrm{B}) \subseteq \mathrm{PFun}(A, \mathrm{B}) \subseteq \mathrm{Rel}(A, \mathrm{B}) \ .$$

Enumerability

Definition 142

1. *A set A is said to be* enumerable *whenever there exists a surjection $\mathbb{N} \twoheadrightarrow A$, referred to as an* enumeration*.*

2. *A* countable *set is one that is either empty or enumerable.*

An enumeration of a set A is a surjection

$e : \mathbb{N} \twoheadrightarrow A$

Idea: $e(0), e(1), e(2) \ldots, e(n), \ldots$ $n \in \mathbb{N}$

lists

$A = \{ e(n) \mid n \in \mathbb{N} \}$

Examples:

1. A bijective enumeration of $\mathbb{Z}$.

$\cdots$	-3	-2	-1	0	1	2	3	$\cdots$
	$\ldots$	3	1	0	2	4	$\ldots$	

$e : \mathbb{N} \twoheadrightarrow \mathbb{Z}$ $\quad e(n) = (-1)^{n \underline{\text{ mod }} 2} \left((n+1) \underline{\text{ div }} 2 \right)$

$e : \mathbb{N} \twoheadrightarrow \mathbb{N} \times \mathbb{N}$ $\qquad$ $e(n) = (k, \ell - k)$

2. A bijective enumeration of $\mathbb{N} \times \mathbb{N}$.

where $n = \dfrac{\ell(\ell+1)}{2} + k$

for $0 \leq k \leq \ell$

	0	1	2	3	4	5	$\cdots$
0	0	1	3				
1	2	4					
2	5						
3							
4							
$\vdots$							

Exercise: Show that e is surjective.

Proposition 143 *Every non-empty subset of an enumerable set is enumerable.*

PROOF: Let A be a set and let $e : \mathbb{N} \twoheadrightarrow A$.

Consider $S \subseteq A$ non-empty, say $x \in S$.

<u>RTP</u>: S is enumerable. That is, there is a surjection $f : \mathbb{N} \twoheadrightarrow S$.

Let

$$f(n) =_{\text{def}} \begin{cases} e(n) & \text{if } e(n) \in S \\ x & \text{if } e(n) \notin S \end{cases}$$

Exercise: Show that f is surjective. $\boxtimes$

Countability

Proposition 144

1. $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ *are countable sets.*

2. *The product and disjoint union of countable sets is countable.*

3. *Every finite set is countable.*

4. *Every subset of a countable set is countable.*

Proposition: The product of enumerable sets is enumerable

PROOF: Let $e_1 : \mathbb{N} \twoheadrightarrow A_1$ and $e_2 : \mathbb{N} \twoheadrightarrow A_2$ be surjections. We show that there is a surjection $\mathbb{N} \twoheadrightarrow A_1 \times A_2$. Consider $e_1 \times e_2 : \mathbb{N} \times \mathbb{N} \rightarrow A_1 \times A_2$ given by $(e_1 \times e_2)(i, j) = (e_1(i), e_2(j))$. Exercise: Show that $e_1 \times e_2$ is a surjection. Then take a surjection $\mathbb{N} \twoheadrightarrow \mathbb{N} \times \mathbb{N}$ and obtain a surjection $\mathbb{N} \twoheadrightarrow \mathbb{N} \times \mathbb{N} \twoheadrightarrow A_1 \times A_2$.

$\boxtimes$

Corollary: $\mathbb{Q}$ is enumerable

PROOF: $\mathbb{Z}$ is enumerable

$\mathbb{N}^+$ is enumerable

$\mathbb{Z} \times \mathbb{N}^+$ is enumerable

$$\mathbb{N} \twoheadrightarrow \mathbb{Z} \times \mathbb{N}^+ \twoheadrightarrow \mathbb{Q}$$

$$(n, m) \longmapsto n/m$$

$\boxtimes$

Proposition: The disjoint union of enumerable sets is enumerable

PROOF: Let $e_1 : \mathbb{N} \twoheadrightarrow A_1$ and $e_2 : \mathbb{N} \twoheadrightarrow A_2$ be surjections. We need construct a surjection $\mathbb{N} \twoheadrightarrow A_1 \uplus A_2$. First, consider

$$e : \mathbb{N} \twoheadrightarrow \mathbb{N} \uplus \mathbb{N}$$
$$n \longmapsto (\underline{\text{rem}}(n,2), \underline{\text{quo}}(n,2))$$

Exercise: Show e is surjective.

Second, consider

$$\mathbb{N} \uplus \mathbb{N} \twoheadrightarrow A_1 \uplus A_2$$

defined by

$$e_1 \uplus e_2 : \mathbb{N} \uplus \mathbb{N} \to A_1 \uplus A_2$$

$$(0, n) \longmapsto (0, e_1(n))$$

$$(1, n) \longmapsto (1, e_2(n))$$

Exercise: Show that $e_1 \uplus e_2$ is a surjection.

Then

$$\begin{tikzcd} \mathbb{N} \arrow[r, two heads, "e"] \arrow[rr, bend right, two heads] & \mathbb{N} \uplus \mathbb{N} \arrow[r, two heads, "e_1 \uplus e_2"] & A_1 \uplus A_2 \end{tikzcd}$$

$\boxtimes$