Finite cardinality

Definition 136 *A set* A *is said to be* finite *whenever* $A \cong [n]$ *for some* $n \in \mathbb{N}$*, in which case we write* $\#A = n$*.*

Theorem 137 *For all* $m, n \in \mathbb{N}$,

1. $\mathcal{P}([n]) \cong [2^n]$
2. $[m] \times [n] \cong [m \cdot n]$
3. $[m] \uplus [n] \cong [m + n]$
4. $([m] \rightharpoonup [n]) \cong [(n+1)^m]$
5. $([m] \Rightarrow [n]) \cong [n^m]$
6. $\mathrm{Bij}([n],[n]) \cong [n!]$

For $m, n \in \mathbb{N}$

(i) $[m] \times [n] \cong [m \cdot n]$

(ii) $[m] \uplus [n] \cong [m+n]$

(i) Consider

$$\begin{aligned} [m] \times [n] &\longrightarrow [m \cdot n] \\ (q, r) &\longmapsto q \cdot n + r \end{aligned}$$

and show it is a bijection

(iii) Consider

$$\begin{aligned} [m] \uplus [n] &\longrightarrow [m+n] \\ (0,i) &\longmapsto i \\ (1,j) &\longmapsto m+j \end{aligned}$$

and show it is a bijection.

$$\forall m \in \mathbb{N}.\ \forall n \in \mathbb{N}.\ ([m] \Rightarrow [n]) \cong [n^m]$$

PROOF: By induction on $m \in \mathbb{N}$.

BASE CASE:

$$\forall n \in \mathbb{N}.\ ([0] \Rightarrow [n]) \cong [n^0]$$

Now, $(\emptyset \Rightarrow X)$ is a singleton inhabited by the unique function $\emptyset \to X$; namely, given by the empty relation.

Also, $[n^0] = [1]$. And we are done.

INDUCTIVE STEP:

Assume

(IH) $\forall n \in \mathbb{N}. ([m] \Rightarrow [n]) \cong [n^m]$

for $m \in \mathbb{N}$.

RTP:

$\forall n \in \mathbb{N}. ([m+1] \Rightarrow [n]) \cong [n^{m+1}]$

Let $n \in \mathbb{N}$.

Then $([m+1] \Rightarrow [n]) \cong (([m] \uplus [1]) \Rightarrow [n])$

$$\big(([m] \uplus [1]) \Rightarrow [n]\big) \cong ([m] \Rightarrow [n]) \times ([1] \Rightarrow [n])$$

$$\cong ([m] \Rightarrow [n]) \times [n]$$

$$\cong [n^m] \times [n] \qquad \text{, by (IH)}$$

$$\cong [n^m \times n]$$

$$= [n^{m+1}]$$

$\boxtimes$

Infinity axiom

$\boxed{\text{There is an infinite set, containing } \emptyset \text{ and closed under successor.}}$