Calculus of bijections

- $A \cong A \;,\; A \cong B \implies B \cong A \;,\; (A \cong B \wedge B \cong C) \implies A \cong C$
- If $A \cong X$ and $B \cong Y$ then

$$\mathcal{P}(A) \cong \mathcal{P}(X) \quad , \quad A \times B \cong X \times Y \quad , \quad A \uplus B \cong X \uplus Y \quad ,$$

$$\mathrm{Rel}(A,B) \cong \mathrm{Rel}(X,Y) \quad , \quad (A \rightharpoonup B) \cong (X \rightharpoonup Y) \quad ,$$

$$(A \Rightarrow B) \cong (X \Rightarrow Y) \quad , \quad \mathrm{Bij}(A,B) \cong \mathrm{Bij}(X,Y)$$

Proposition: For sets A, B, X, Y,

$$A \cong X \wedge B \cong Y \implies \underline{Rel}(A,B) \cong \underline{Rel}(X,Y)$$

PROOF: Let $A \cong X$, with $f : A \to X$ a bijection.
Let $B \cong Y$, with $g : B \to Y$ a bijection.

The mapping

$$\begin{aligned} \underline{Rel}(A,B) &\longrightarrow \underline{Rel}(X,Y) \\ R &\longmapsto g \circ R \circ f^{-1} \end{aligned}$$

is a function

It is a bijection with inverse

$$\begin{aligned} \underline{\mathrm{Rel}}(X,Y) &\longrightarrow \underline{\mathrm{Rel}}(A,B) \\ S &\longmapsto g^{-1}\circ S\circ f \end{aligned}$$

For $R \in \underline{\mathrm{Rel}}(A,B)$

$$R \longmapsto g\circ R\circ f^{-1} \longmapsto g^{-1}\circ g\circ R\circ f^{-1}\circ f = R$$

$$
\begin{tikzcd}
 & S \in \underline{Rel}(X,Y) \arrow[dl, mapsto] \arrow[d, equal] \\
g^{-1} \circ S \circ f \arrow[r, mapsto] & g \circ g^{-1} \circ S \circ f \circ f^{-1}
\end{tikzcd}
$$

$\boxtimes$

Arithmetic Laws

Recall that for finite sets A and B,

$\#(A \times B) = \#(A) \cdot \#(B)$ multiplication

$\#(A \uplus B) = \#(A) + \#(B)$ addition

$\#(A \Rightarrow B) = (\#B)^{\#(A)}$ exponentiation

- The arithmetic laws have set-theoretic counterparts

Eg: $(a+b) \cdot c = a \cdot c + b \cdot c$ $\quad (A \uplus B) \times C \cong (A \uplus C) \times (B \uplus C)$

- $A \cong [1] \times A$, $(A \times B) \times C \cong A \times (B \times C)$, $A \times B \cong B \times A$
- $[0] \uplus A \cong A$, $(A \uplus B) \uplus C \cong A \uplus (B \uplus C)$, $A \uplus B \cong B \uplus A$
- $[0] \times A \cong [0]$, $(A \uplus B) \times C \cong (A \times C) \uplus (B \times C)$
- $(A \Rightarrow [1]) \cong [1]$, $(A \Rightarrow (B \times C)) \cong (A \Rightarrow B) \times (A \Rightarrow C)$
- $([0] \Rightarrow A) \cong [1]$, $((A \uplus B) \Rightarrow C) \cong (A \Rightarrow C) \times (B \Rightarrow C)$
- $([1] \Rightarrow A) \cong A$, $((A \times B) \Rightarrow C) \cong (A \Rightarrow (B \Rightarrow C))$
- $(A \rightharpoonup B) \cong (A \Rightarrow (B \uplus [1]))$
- $\mathcal{P}(A) \cong (A \Rightarrow [2])$

Arithmetic-like Bijections

$$(A \uplus B) \times C \cong (A \times C) \uplus (B \times C)$$

<u>Proposition</u> Let X, Y, Z be sets.
If X and Y are disjoint then
(i) $X \times Z$ and $Y \times Z$ are disjoint
(ii) $(X \cup Y) \times Z \cong (X \times Z) \cup (Y \times Z)$

PROOF: Let X, Y, Z be sets.

Assume that X and Y are disjoint; that is, $X \cap Y = \emptyset$.

(i) <u>RTP</u>: $X \times Z$ and $Y \times Z$ are disjoint.

Suppose that $(X \times Z) \cap (Y \times Z)$ is inhabited. That is, there exists an element, say (a, z) in both $X \times Z$ and $Y \times Z$. Therefore, there exists an element, namely a, both in X and in Y. A contradiction. Hence,

$$(X \times Z) \cap (Y \times Z) = \emptyset.$$

(ii) The mapping

$$(X \cup Y) \times Z \longrightarrow (X \times Z) \cup (Y \times Z)$$

$$(a, z) \longmapsto (a, z)$$

is a bijective function. Exercise

$\boxtimes$

$$c^{a \cdot b} = (c^b)^a$$

$$((A \times B) \Rightarrow C) \cong (A \Rightarrow (B \Rightarrow C))$$

In OCaml notation:

<u>curry</u>$(f) =$ fun $a \to$ fun $b \to f(a,b)$

of type $(\alpha * \beta \to \gamma) \to (\alpha \to \beta \to \gamma)$

<u>uncurry</u>$(h) =$ <u>fun</u> $(a,b) \to h\ a\ b$

of type $(\alpha \to \beta \to \gamma) \to (\alpha * \beta \to \gamma)$

Exercise: Show that <u>curry</u> and <u>uncurry</u> are inverses of each other.

$$c^{a+b} = c^a \cdot c^b$$

$$((A \uplus B) \Rightarrow C) \cong (A \Rightarrow C) \times (B \Rightarrow C)$$

Consider the mapping

$$(A \uplus B \Rightarrow C) \longrightarrow (A \Rightarrow C) \times (B \Rightarrow C)$$

$$(f : A \uplus B \to C) \longmapsto (f_0, f_1)$$

where $f_0 : A \to C : a \mapsto f(0, a)$

and $f_1 : B \to C : b \mapsto f(1, b)$

Exercise: Prove that it is a bijection.

Characteristic (or Indicator) Functions

Recall that for finite sets A

$$\#\mathcal{P}(A) = 2^{\#A} = \#(A \Rightarrow [2])$$

In fact,

$$\mathcal{P}(A) \cong (A \Rightarrow [2])$$

for all sets.

PROOF:
$$\mathcal{P}(A) \longrightarrow (A \Rightarrow [2])$$
$$S \subseteq A \longmapsto \chi_S : A \to [2] : a \mapsto \begin{cases} 1, & a \in S \\ 0, & a \notin S \end{cases}$$

$$(A \Rightarrow [2]) \longrightarrow \mathcal{P}(A)$$

$$(f : A \to [2]) \longmapsto \bar{f} \subseteq A$$

$$\bar{f} \stackrel{\text{def}}{=} \{a \in A \mid f(a) = 1\}$$

Exercise: Show that the given functions establish a bijection. $\boxtimes$