Equivalence relations and set partitions

- Equivalence relations.

$R \subseteq A \times A$ is an equivalence relation whenever

(1) Reflexive: $\forall x \in A.\ x R x$

(2) Symmetry: $\forall x, y \in A.\ x R y \Rightarrow y R x$

(3) Transitive: $\forall x, y, z \in A.$

$$x R y \wedge y R z \Rightarrow x R z.$$

Examples:

- For m a positive integer, let $R_m \subseteq \mathbb{Z} \times \mathbb{Z}$

  $x \, R_m \, y \overset{def}{\iff} x \equiv y \pmod{m}$

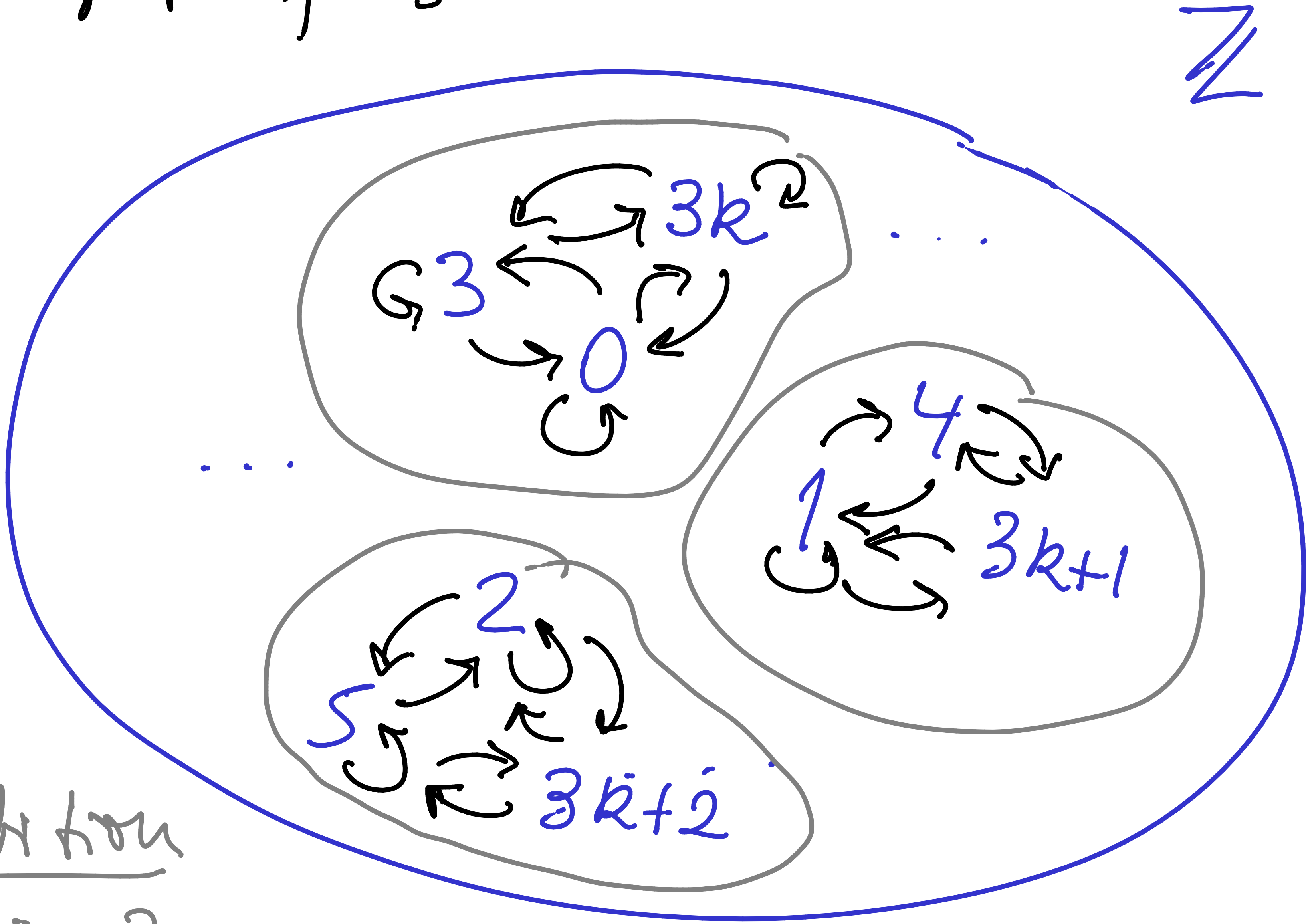
- Let A be a set.

  $I \subseteq \mathcal{P}(A) \times \mathcal{P}(A)$

  $\overset{def}{=}$

  $\{ (U, V) \in \mathcal{P}(A) \times \mathcal{P}(A) \mid U \cong V \}.$

Internal graph of R_3



A partition of $\mathbb{Z}$ in 3 equivalence classes.

- Set partitions.

A partition P of a set A is a set of subsets of A $\quad P \subseteq \mathcal{P}(A)$

such that

(1) $\emptyset \in P$

(2) $\bigcup P = A$

(3) $\forall U, V \in P.\ U \neq V \Rightarrow U \cap V = \emptyset$

Examples: Partitions of $\mathbb{Z}$.

$P_1 = \{ \mathbb{Z} \}$

$P_2 = \{ \text{Odd}, \text{Even} \}$

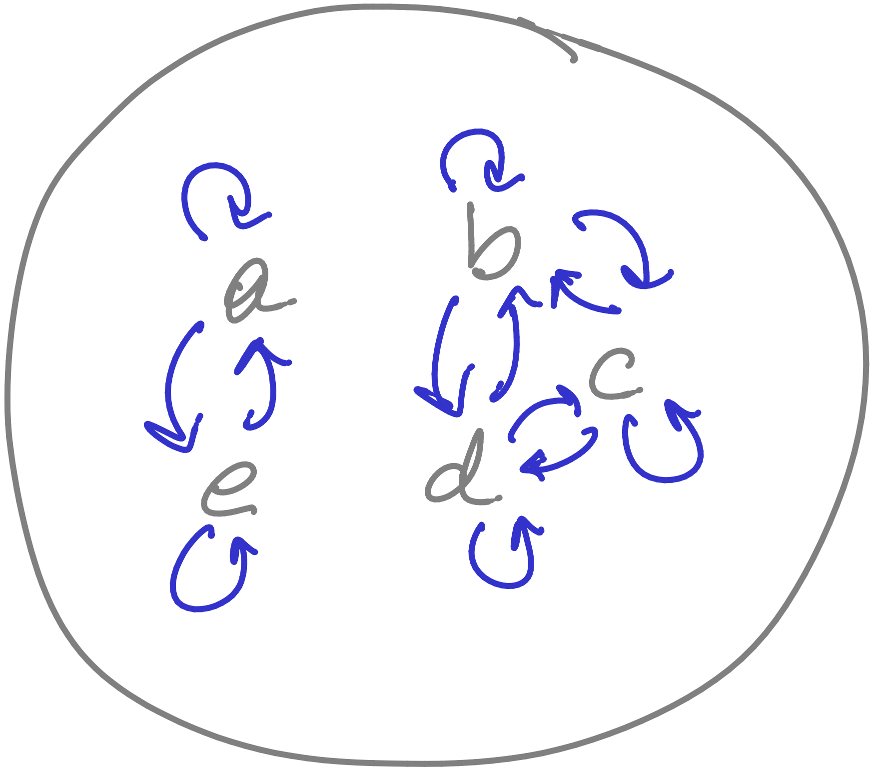
$P_3 = \{ \{3k \mid k \in \mathbb{Z}\}, \{3k+1 \mid k \in \mathbb{Z}\}, \{3k+2 \mid k \in \mathbb{Z}\} \}$

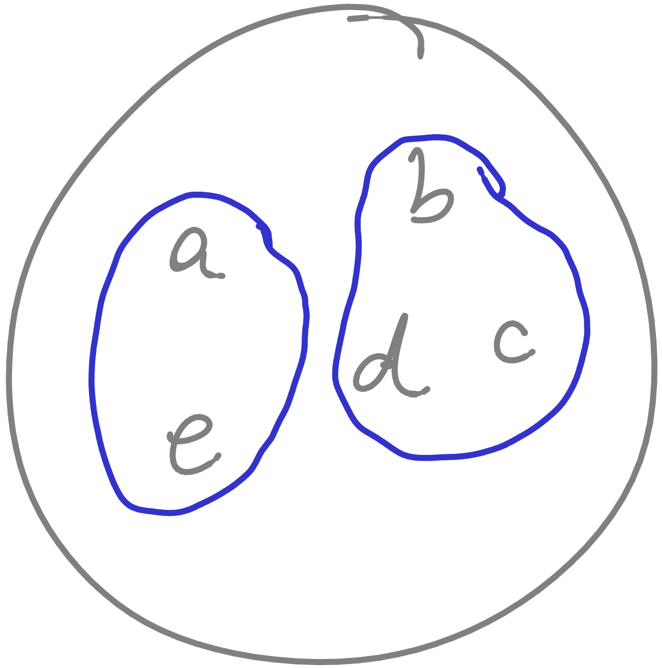
Theorem 134 *For every set* A,

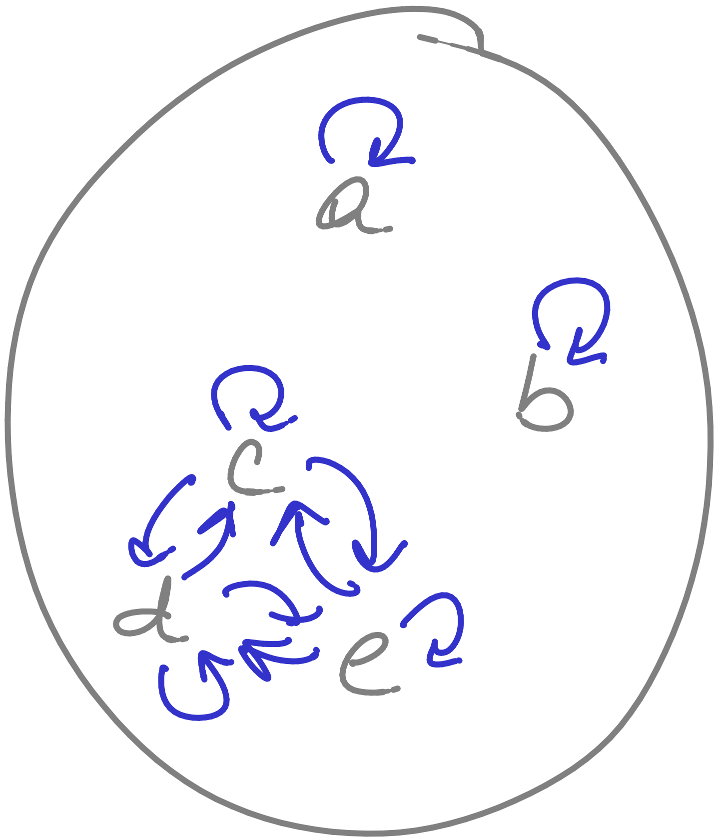
$$\mathrm{EqRel}(A) \cong \mathrm{Part}(A).$$

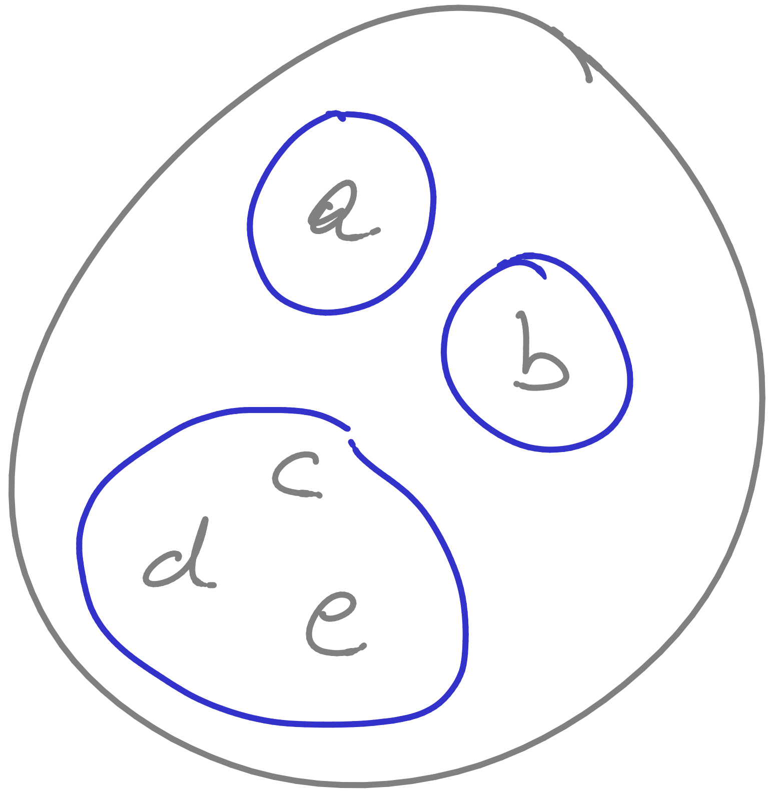
PROOF: all equivalence relations on A

all partitions of A






a
b
c
d
e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$\underline{\text{part}} : \underline{\text{EqRel}}(A) \longrightarrow \underline{\text{Part}}(A)$

$$E \longmapsto \underline{\text{part}}(E)$$

equivalence relation on A

a partition of A.

<u>Def</u> For $a \in A$, let

$[a]_E$ ⟵ The equivalence class of a under E

$\overset{\text{def}}{=}$

$\{x \in A \mid x E a\}$

$\underline{part}(E) \stackrel{def}{=} \{ \beta \subseteq A \mid \exists a \in A.\ \beta = [a]_E \}$

We have a function

$\underline{part} : \underline{EqRel}(A) \to \underline{Part}(A)$

Iff

for all equivalence relations E on A,

$\underline{part}(E)$ is a partition of A.

(1) $\emptyset \notin \underline{part}(E)$

Because every set in $\underline{part}(E)$ is of

The form $[a]_E$ for $a \in A$

and $a \in [a]_E$. Hence $[a]_E \neq \emptyset$.

(2) $\bigcup \underline{\mathrm{part}}(E) = A$

$$\bigcup \{ [a]_E \subseteq A \mid a \in A \} = A$$

Clearly $\bigcup \underline{\mathrm{part}}(E) \subseteq A$. So we show $A \subseteq \bigcup \underline{\mathrm{part}}(E)$. Equivalently, for $x \in A$. $\exists a \in A . x \in [a]_E$; which holds because $x \in [x]_E$.

(3) $\forall \beta_1, \beta_2 \in \underline{Part}(E)$.

$$\beta_1 \cap \beta_2 \neq \emptyset \Rightarrow \beta_1 = \beta_2$$

Let $\beta_1, \beta_2 \in \underline{part}(E)$. Then There exist $a_1, a_2 \in A$ such That $\beta_1 = [a_1]_E$ and $\beta_2 = [a_2]_E$.

Assume

$$[a_1]_E \cap [a_2]_E \neq \emptyset$$

and let $x \in [a_1]_E \cap [a_2]_E$.

Then, $x E a_1$ and $x E a_2$ and Therefore $a_1 E a_2$.

Lemma $a_1 E a_2 \Rightarrow [a_1]_E = [a_2]_E$

Exercise.

Lemma $[a_1]_E = [a_2]_E \Rightarrow a_1 E a_2$

Exercise

- $\underline{eq} : \underline{Part}(A) \longrightarrow \underline{EqRel}(A)$

$$P \longmapsto \underline{eq}(P) \subseteq A \times A$$

s.t. it is an equivalence relation for all partitions P of A.

For $x, y \in A$

$$x \ \underline{eq}(P) \ y \overset{\text{def}}{\iff} \exists \beta \in P.\ x \in \beta \wedge y \in \beta.$$

(1) $\forall x \in A.\ x \ \underline{eq}(P)\ x.$

$\iff \exists \beta \in P.\ x \in \beta.$

which holds because $\bigcup P = A.$

(2) $\forall x, y \in A.\ x \ \underline{eq}(P)\ y \implies y \ \underline{eq}(P)\ x.$

Trivially.

(3) $\forall x, y, z \in A.\ x \ \underline{eq}(P)\ y \wedge y \ \underline{eq}(P)\ z$

$$\implies x \ \underline{eq}(P)\ z.$$

Let $x, y, z \in A$. Assume $x \ \underline{eq}(P)\ y$; that is.

$\exists \beta \in P.\ x, y \in \beta$. Assume $y \ \underline{eq}(P)\ z$; that is

$\exists \gamma \in P.\ y, z \in \gamma.$

We have $x, y \in \beta$ and $y, z \in \gamma$

Therefore $\beta \cap \gamma \neq \emptyset$ Thus $\beta = \gamma$

Hence $x, z \in \beta$. and so

$x \ \underline{eq}(P)\ z.$

$\boxtimes$

$$\begin{tikzcd} \underline{\mathrm{EqRel}}(A) \arrow[r, bend left, "\underline{\mathrm{part}}"] \arrow[r, phantom, "\cong"] & \underline{\mathrm{Part}}(A) \arrow[l, bend left, "\underline{\mathrm{eq}}"] \end{tikzcd}$$

iff

(1) $\forall E \in \underline{\mathrm{EqRel}}(A).$

$$\underline{\mathrm{eq}}(\underline{\mathrm{part}}(E)) = E$$

(2) $\forall P \in \underline{\mathrm{Part}}(A).$

$$\underline{\mathrm{part}}(\underline{\mathrm{eq}}(P)) = P$$

(1) Let $E \in \underline{EqRel}(A)$

$\underline{eq}(\underline{part}(E))$

$= \{(x,y) \in A \times A \mid \exists \beta \in \underline{part}(E).\ x,y \in \beta\}$

$= \{(x,y) \in A \times A \mid \exists a \in A.\ x,y \in [a]_E\}$

$= \{(x,y) \in A \times A \mid x E y\}$

$= E$

(2) Let $P \in \underline{Part}(A)$

Consider

$\underline{part}(\underline{eq}(P))$

$= \{ \alpha \subseteq A \mid \exists a \in A.\ \alpha = [a]_{\underline{eq}(P)} \}$

Since P is a partition, for every $a \in A$, there exists a unique $B(a) \in P$ such that $a \in B(a)$

Then, $[a]_{\underline{eq}(P)} = \{ x \in A \mid \exists \beta \in P.\ x \in \beta \wedge a \in \beta \}$

$= \{ x \in A \mid x \in B(a) \} = B(a)$

Hence

$$\underline{\mathrm{part}}(\underline{\mathrm{eq}}(P))$$

$$= \{ \alpha \subseteq A \mid \exists a \in A.\ \alpha = B(a) \}$$

Moreover

$$\alpha \in P \iff \exists a \in A.\ \alpha = B(a)$$

Therefore

$$\underline{\mathrm{part}}(\underline{\mathrm{eq}}(P)) = \{ \alpha \subseteq A \mid \alpha \in P \} = P$$

$$\boxtimes$$