Bijections — invertible or reversible

$f: A \to B$ is a bijection whenever there is $l: B \to A$ and $r: B \to A$ such that

$$l \circ f = \mathrm{id}_A \quad \text{and} \quad f \circ r = \mathrm{id}_B$$

NB: In this case,

$$l = r.$$

lofor

rdor lord

r l

NB: Inverses of bijections are unique
The inverse of f is denoted f^{-1}.

Bijections

Definition 127 *A function $f : A \to B$ is said to be* bijective*, or a* bijection*, whenever there exists a (necessarily unique) function $g : B \to A$ (referred to as the* inverse *of f) such that*

1. *g is a* retraction *(or* left inverse*) for f:*

$$g \circ f = \mathrm{id}_A \quad ,$$

2. *g is a* section *(or* right inverse*) for f:*

$$f \circ g = \mathrm{id}_B \quad .$$

Examples

- $$\begin{tikzcd} \underline{\mathrm{Rel}}([n],[n]) \arrow[r, bend left, "\underline{\mathrm{mat}}"] & \text{Boolean } (n \times n)\text{-matrices} \arrow[l, bend left, "\underline{\mathrm{rel}}"] \end{tikzcd}$$

- $$\begin{tikzcd} \mathbb{Z} \arrow[r, bend left, "\underline{\mathrm{succ}}"] & \mathbb{Z} \arrow[l, bend left, "\mathrm{pred}"] \end{tikzcd}$$

$$\underline{\mathrm{succ}}(n) = n+1$$
$$\underline{\mathrm{pred}}(n) = n-1$$

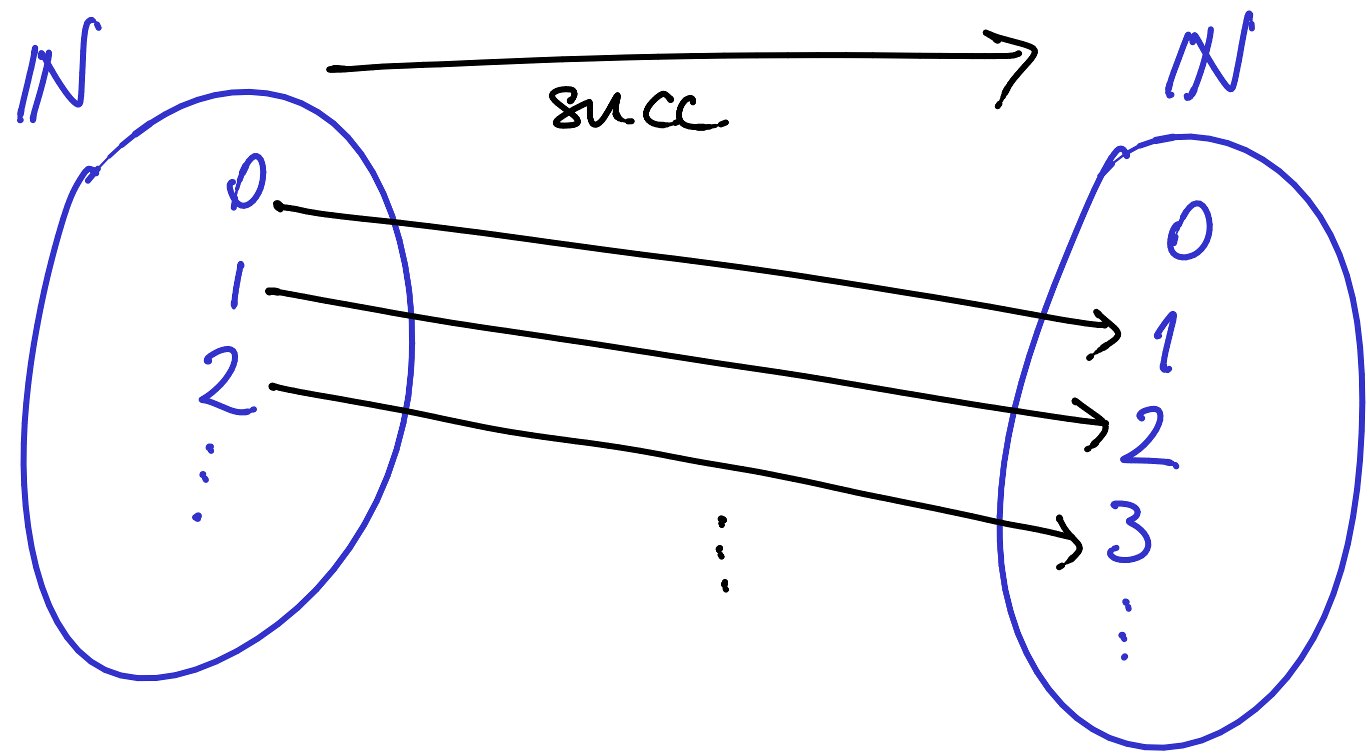
Non-example

$$\begin{tikzcd} \mathbb{N} \arrow[r, "succ"] & \mathbb{N} \arrow[l, bend left, "g"] \end{tikzcd}$$

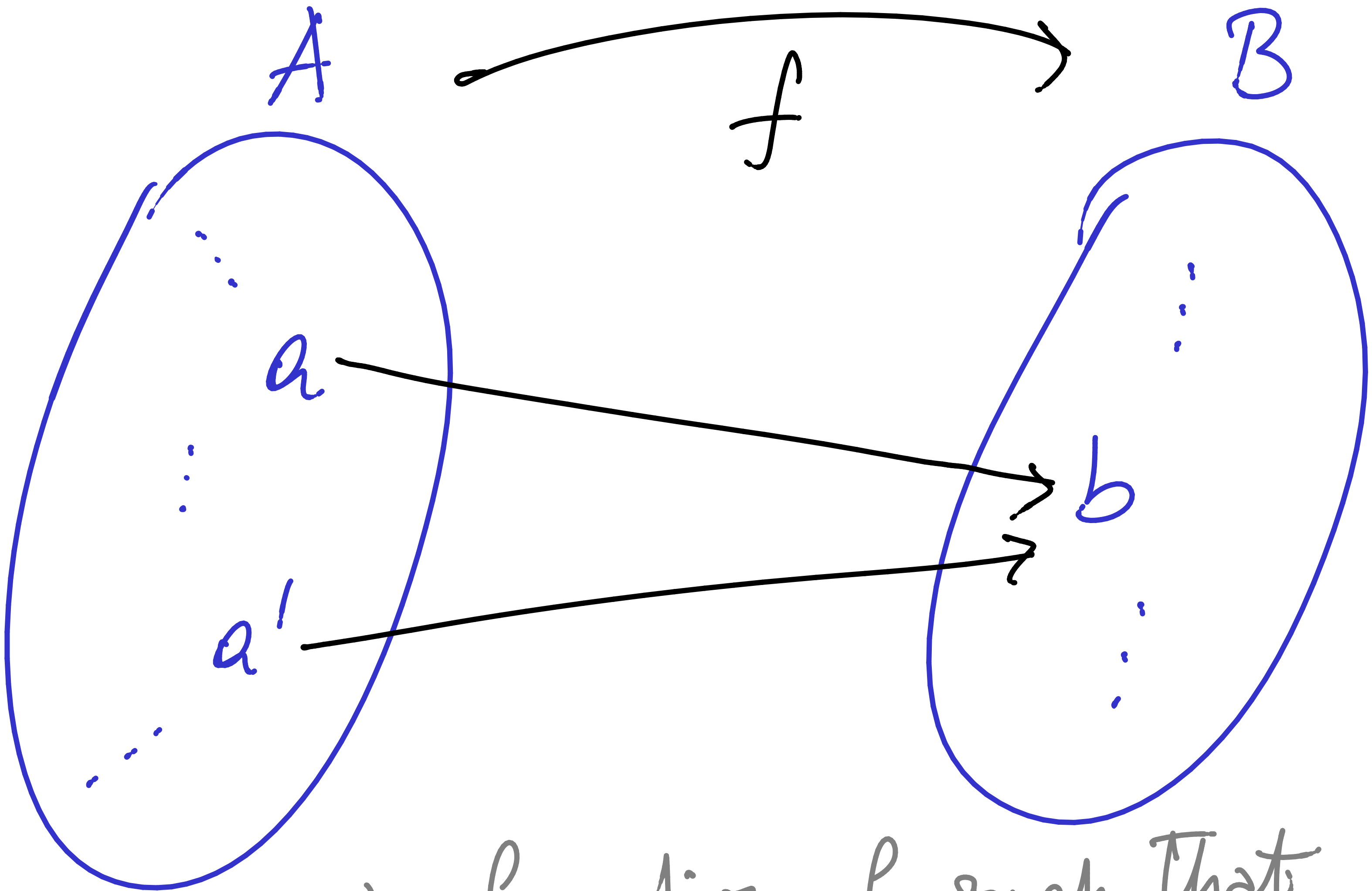
is not a bijection

$$\begin{tikzcd} & n \in \mathbb{N} \arrow[dl, mapsto] \\ g(n) \arrow[r, mapsto, "\underline{succ}"'] & g(n)+1 \end{tikzcd}$$

$n \overset{?}{=} g(n)+1 \implies g(0)+1 = 0$ ↯



There is $k \in \mathbb{N}$, namely $k = 0$, such that, for all $n \in \mathbb{N}$, $\underline{\mathrm{succ}}(n) \neq k$.



a function f such that $f(a) = f(a')$ for $a \neq a'$ is not a bijection.

Proposition: A function $f: A \to B$ is a bijection if, and only if,

$$\forall b \in B.\ \exists!\, a \in A.\ f(a) = b.$$

PROOF: ($\Longrightarrow$) Assume there exists $g: B \to A$ such that $f \circ g = id_B$ and $g \circ f = id_A$.

<u>RTP</u>: $\forall b \in B.\ \exists!\, a \in A.\ f(a) = b.$

Let $b \in B$. Then $g(b) \in A$ is such that $f(g(b)) = b$. Therefore $\exists a \in A.\ f(a) = b$.

We need show uniqueness.

<u>RTP</u>: $\forall a, a' \in A.\ f(a) = b \wedge f(a') = b \Rightarrow a = a'$

Let $a, a' \in A$. Assume $f(a) = b$ and $f(a') = b$.

Then

$$f(a) = f(a')$$

and therefore

$$a = g(f(a)) = g(f(a')) = a'$$

($\Leftarrow$) Assume

$$\forall b \in B.\ \exists! a \in A.\ f(a) = b.$$

<u>RTP</u>: f has both a left and a right inverse, say $g : B \to A$.

Define

$$g : \underset{\substack{\uparrow \\ B}}{b} \longmapsto \text{The unique } a \in A \text{ such that } f(a) = b.$$

Then,

(1) g is a total function from B to A.

(2) For $b \in B$, $f(g(b)) = b$

$$f \circ g = id_B.$$

(3) For $a \in A$,

$g(f(a))$ is the unique $x \in A$ s.t. $f(x) = f(a)$

and $f(a) = f(a)$

therefore $g(f(a)) = a$

Hence, $g \circ f = id_A$ $\boxtimes$

Proposition 129 *For all finite sets* A *and* B,

$$\#\,\mathrm{Bij}(A,B) = \begin{cases} 0 & \text{, if } \#A \neq \#B \\ n! & \text{, if } \#A = \#B = n \end{cases}$$

PROOF IDEA: $A = \{a_1, \ldots, a_m\}$ $B = \{b_1, \ldots, b_n\}$

There is a bijection between A and B

iff $\forall\, b_j\ (1 \leq j \leq n)\ \exists!\, a_i\ (1 \leq i \leq m)$.

iff $m = n$.

Let $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_n\}$.

$a_1 \longmapsto b_{j_1}$ n- choices

$a_2 \longmapsto b_{j_2}$ (n-1)- choices $b_{j_2} \neq b_{j_1}$

$a_3 \longmapsto b_{j_3}$ (n-2) choices. $b_{j_3} \neq b_{j_1}$
$\neq b_{j_2}$

$\vdots$

$a_{n-1} \longmapsto b_{j_{n-1}}$ 2- choices

$a_n \longmapsto b_{j_n}$ 1- choice

Number of bijections is $n \times (n-1) \times \dots \times 2 \times 1$
$= n!$ $\boxtimes$

Theorem 130 *The identity function is a bijection, and the composition of bijections yields a bijection.*

NB: $(\mathrm{id}_A)^{-1} = \mathrm{id}_A$

For $f: A \to B$ and $g: B \to C$ bijections, $g \circ f: A \to C$ bijection with inverse

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1} : C \to A.$$

NB: $\left(\underline{\mathrm{Bij}}(A, A), \mathrm{id}_A, \circ \right)$ is a group.

Definition 131 *Two sets* A *and* B *are said to be* isomorphic *(and to have the* same cardinatity*) whenever there is a bijection between them; in which case we write*

$$A \cong B \quad \textit{or} \quad \#A = \#B \quad .$$

Examples:

1. $\{0, 1\} \cong \{\mathbf{false}, \mathbf{true}\}$.

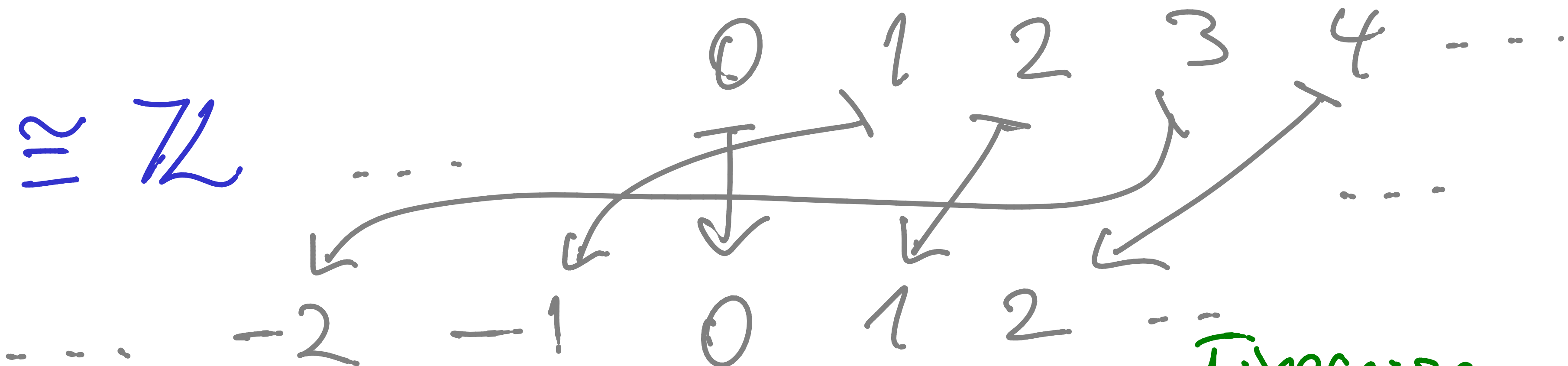
2. $\mathbb{N} \cong \mathbb{N}^+ \;,\; \mathbb{N} \cong \mathbb{Z} \;,\; \mathbb{N} \cong \mathbb{N} \times \mathbb{N} \;,\; \mathbb{N} \cong \mathbb{Q}$.

Examples

$$\mathbb{N} \cong \mathbb{N}^+$$

$$\begin{tikzcd} \mathbb{N} \arrow[r, bend left, "\text{succ}"] & \mathbb{N}^+ \arrow[l, bend left, "\text{pred}"] \end{tikzcd}$$

$$\mathbb{N} \cong \mathbb{Z}$$



$$\begin{tikzcd} \mathbb{N} \arrow[r, bend left, "f"] & \mathbb{Z} \arrow[l, bend left, "g"] \end{tikzcd}$$

$$f(n) = \begin{cases} k & , \text{ if } n = 2k \\ -(k+1), & \text{if } n = 2k+1 \end{cases}$$

Exercise: Define g such that $f \circ g = id_{\mathbb{Z}}$ and $g \circ f = id_{\mathbb{N}}$