Functions

$$(A \Rightarrow B) \subseteq (A \rightrightarrows B) \subseteq \underline{\mathrm{Rel}}(A, B)$$

The set of all functions from A to B

Functions (or maps)

Definition 123 *A partial function is said to be* total*, and referred to as a* (total) function *or* map*, whenever its domain of definition coincides with its source.*

$f : A \to B$ is a (total) function whenever $\underline{\mathrm{dom}}(f) = A$.

equivalently $\forall a \in A .\ f(a) \downarrow$

Theorem 124 *For all* $f \in \mathrm{Rel}(A, B)$,

$$f \in (A \Rightarrow B) \iff \forall\, a \in A .\ \exists!\, b \in B .\ a\, f\, b \quad .$$

Example: Total predecessor function.

$$\underline{\text{totpred}} : \mathbb{N} \to \mathbb{N}$$

$$\underline{\text{totpred}}(n) = \begin{cases} 0 & \text{if } n = 0 \\ n-1 & \text{if } n \geq 1 \end{cases}$$

Inductive Definitions

Example:

$\underline{add} : \mathbb{N} \to \mathbb{N}$

$$\begin{cases} \underline{add}(m,0) \overset{def}{=} m \\ \underline{add}(m,n+1) \overset{def}{=} \underline{add}(m,n)+1 \end{cases}$$

Example: $t : \mathbb{N} \to \mathbb{N}$

$t(n) = \sum_{i=0}^{n} i$.

$$\begin{cases} t(0) = 0 \\ t(n+1) = \underline{add}(n, t(n)) \end{cases}$$

Inductive Definitions

The function

$$r : \mathbb{N} \to A$$

inductively defined from

$$a \in A$$

$$f : \mathbb{N} \times A \to A$$

is the unique such that

$$\begin{cases} r(0) = a \\ r(n+1) = f(n, r(n)) & n \in \mathbb{N} \end{cases}$$

Let A be a set. For $a \in A$ and a function $f : \mathbb{N} \times A \to A$.

Define

$$\mathcal{C} =^{def} \{ R \subseteq \mathbb{N} \times A \mid R \text{ is } (a,f)\text{-closed} \}$$

<u>Def</u>: R is (a,f)-closed

iff $0 \, R \, a$

and

$\forall n \in \mathbb{N}, \forall a \in A. \; n \, R \, a \Rightarrow (n+1) \, R \, f(n,a)$

Theorem

(1) The relation

$$r =_{def} \bigcap \mathcal{C} : \mathbb{N} \nrightarrow A$$

is functional and total

(2) The function $r : \mathbb{N} \to A$ is the unique such that

$$r(0) = a$$

and

$$r(n+1) = f(n, r(n)) \text{ for all } n \in \mathbb{N}.$$

Proposition 125 *For all finite sets* A *and* B,

$$\#(A \Rightarrow B) = \#B^{\#A} .$$

PROOF IDEA: $A = \{a_1, \ldots, a_m\}$ $B = \{b_1, \ldots, b_n\}$

$$\left.\begin{array}{lll} a_1 \mapsto b_{j_1} & & n \text{ choices} \\ a_2 \mapsto b_{j_2} & \times & n \text{ choices} \\ \vdots & & \\ a_i \mapsto b_{j_i} & \times & n \text{ choices} \\ \vdots & & \\ a_m \mapsto b_{j_m} & \times & n \text{ choices} \end{array}\right] \quad n^m \text{ choices}$$

$\boxtimes$

EXTENSIONALITY PRINCIPLE

Theorem 126 *The identity partial function is a function, and the composition of functions yields a function.*

two functions are equal if they give the same output on all inputs

NB

1. $f = g : A \to B$ iff $\forall\, a \in A.\, f(a) = g(a)$.

2. For all sets A, the identity function $\mathrm{id}_A : A \to A$ is given by the rule

$$\mathrm{id}_A(a) = a$$

and, for all functions $f : A \to B$ and $g : B \to C$, the composition function $g \circ f : A \to C$ is given by the rule

$$(g \circ f)(a) = g\big(f(a)\big) \quad .$$

$$(f \circ g) \circ h = f \circ (g \circ h)$$

iff

$$\forall x.\ ((f \circ g) \circ h)(x) = (f \circ (g \circ h))(x)$$

$$((f \circ g) \circ h)(x) = (f \circ g)(h(x)) = f(g(h(x)))$$

$$(f \circ (g \circ h))(x) = f((g \circ h)(x)) = f(g(h(x)))$$

$$id \circ f = f$$

iff

$$(id \circ f)(x) = f(x) \quad \forall x$$

$$(id \circ f)(x) = id(f(x))$$