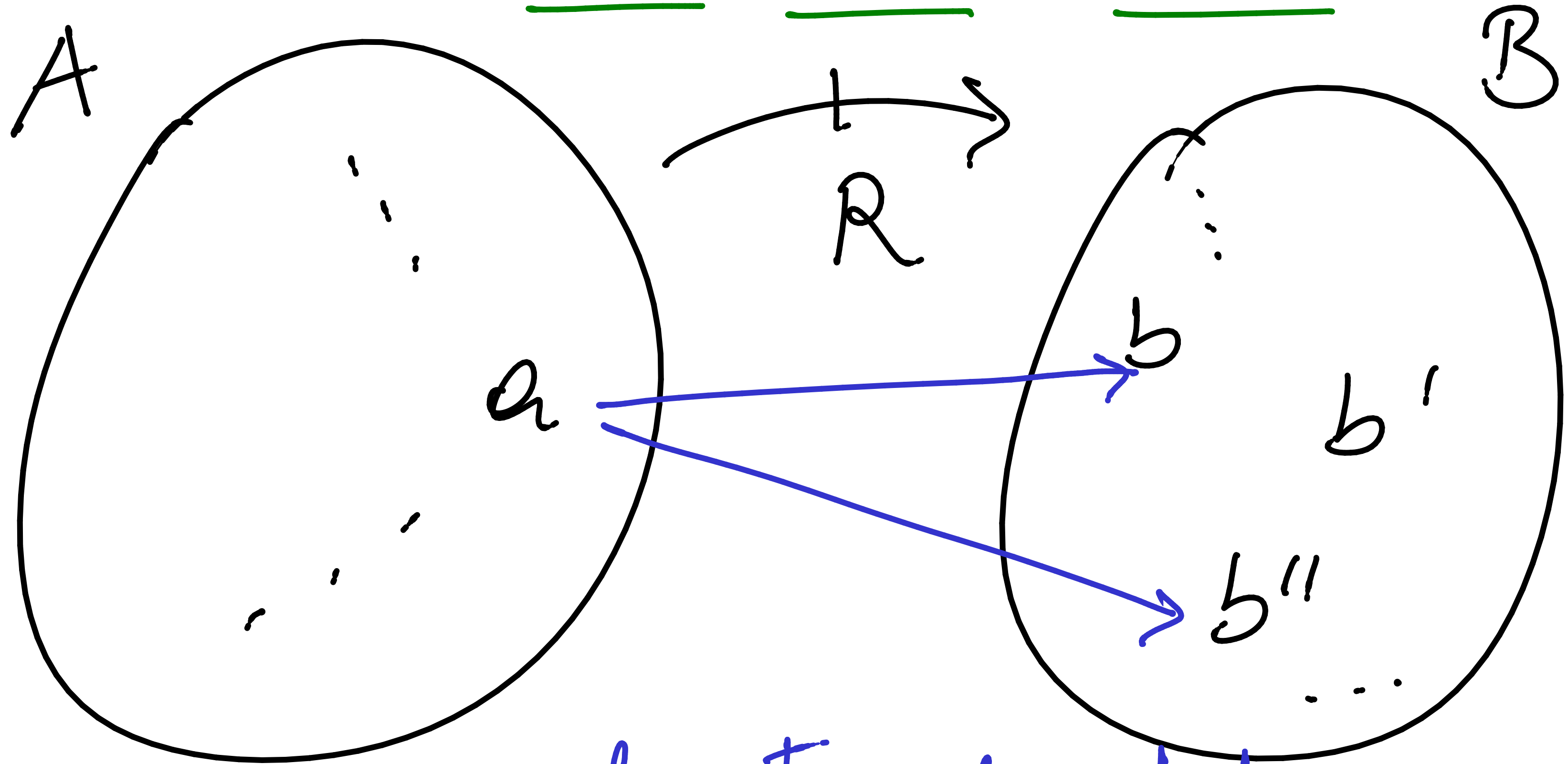
Partial functions

Definition 119 *A relation* $R : A \mathrel{-\!\!\!+\!\!\!\!\rightarrow} B$ *is said to be* <u>functional</u>, *and called a* <u>partial function</u>, *whenever it is such that*

$$\forall a \in A. \forall b_1, b_2 \in B. \ \underline{a \, R \, b_1} \ \wedge \ \underline{a \, R \, b_2} \implies \underline{b_1 = b_2} \quad .$$



a non-functional relation

Notation:

$f : A \to B$ f is a partial function from A to B

Given $a \in A$, we have

either (i) There is no $b \in B$ such that $a f b$

or (ii) There is a unique $b \in B$ such that $a f b$

In case (i), we write

$f(a)\uparrow$ $\quad$ f is undefined at a

In case (ii), we write

$f(a)\downarrow$ $\quad$ f is defined at a

Moreover,

$f(a)$ denotes the unique element of B such that $(a, f(a))$ is in f.

Domain of definition

For $f : A \to B$,

$$\underline{\mathrm{dom}}(f) \subseteq A$$

$$\overset{\mathrm{def}}{=}$$

$$\{ a \in A \mid f(a)\downarrow \} = \{ a \in A \mid \exists b \in B.\ a f b \}.$$

Example:

$$\underline{\text{pred}} : \mathbb{N} \to \mathbb{N}$$

$$\underline{\text{pred}} \overset{\text{def}}{=} \{(n+1, n) \in \mathbb{N} \times \mathbb{N} \mid n \in \mathbb{N}\}$$

$$= \{(x, y) \in \mathbb{N} \times \mathbb{N} \mid x \geq 1 \wedge x = y + 1\}$$

- $\underline{\text{pred}}(0)\uparrow$
- $\underline{\text{pred}}(m)\downarrow$ for m a positive integer

  $\underline{\text{pred}}(m) = m - 1$

- $\underline{\text{dom}}(\underline{\text{pred}})$ is the set of positive integers

Defining partial functions

$$f : A \longrightarrow B$$

$$f : \underset{\underset{A}{\cap}}{a} \longmapsto \underbrace{\cdots\cdots a \cdots\cdots}_{\in B}$$

rule,
mapping,
assignment,
definition,
construction,
etc.

Example:

$$\underline{\text{pred}} : \mathbb{N} \longrightarrow \mathbb{N}$$

$$\underline{\text{pred}} : \underset{\substack{\cap \\ \mathbb{N}}}{n} \longmapsto \max_{k \in \mathbb{N}} \; k < n$$

as a relation:

$$\underline{\text{pred}} = \{ (n,m) \in \mathbb{N} \times \mathbb{N} \mid m = \max_{k \in \mathbb{N}} \; k < n \}$$

For all $n \in \mathbb{N}$ there is at most one element equal to $\max_{k \in \mathbb{N}} k < n$. For $n = 0$ there is no such element, for $n \geq 1$ that element is $n-1$.

Example: Quotient with remainder for integers

$$qr : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{N}$$

$$\underline{\mathrm{dom}}(qr) = \{ (n, m) \in \mathbb{Z} \times \mathbb{Z} \mid m \neq 0 \}$$

$$qr : (n, m) \mapsto (q, r) \in \mathbb{Z} \times \mathbb{N}$$

such that $n = q \cdot m + r$

with $0 \leq r < m$

Example: The following defines a partial function $\mathbb{Z} \times \mathbb{Z} \rightharpoonup \mathbb{Z} \times \mathbb{N}$:

- for $n \geq 0$ and $m > 0$,
  $(n, m) \mapsto \big(\mathrm{quo}(n, m) \,,\, \mathrm{rem}(n, m) \big)$
- for $n \geq 0$ and $m < 0$,
  $(n, m) \mapsto \big(-\mathrm{quo}(n, -m) \,,\, \mathrm{rem}(n, -m) \big)$
- for $n < 0$ and $m > 0$,
  $(n, m) \mapsto \big(-\mathrm{quo}(-n, m) - 1 \,,\, \mathrm{rem}(m - \mathrm{rem}(-n, m), m) \big)$
- for $n < 0$ and $m < 0$,
  $(n, m) \mapsto \big(\mathrm{quo}(-n, -m) + 1 \,,\, \mathrm{rem}(-m - \mathrm{rem}(-n, -m), -m) \big)$

Its domain of definition is $\{ \, (n, m) \in \mathbb{Z} \times \mathbb{Z} \mid m \neq 0 \, \}$.

Notation:

The set of all relations from A to B

$$(A \rightharpoonup B) \subseteq \underline{Rel}(A,B) = \mathcal{P}(A \times B)$$

The set of all partial functions from A to B

- $f = g : A \rightharpoonup B$

Iff $\forall a \in A. \left(f(a)\downarrow \iff g(a)\downarrow \right)$

$\wedge \left[f(a)\downarrow \wedge\, g(a)\downarrow \Rightarrow f(a) = g(a) \right]$

Identities and Composition

- $\mathrm{id}_A \in \underline{\mathrm{Rel}}(A,A)$
  is a partial function $A \to A$

- Let $f : A \to B$ and $g : B \to C$.
  Consider $g \circ f \in \underline{\mathrm{Rel}}(A,C)$

$$g \circ f \overset{\text{def}}{=} \{(a,c) \in A \times C \mid \exists b \in B.\ a\, f\, b \wedge b\, g\, c\}$$

Proposition For partial functions $f : A \to B$ and $g : B \to C$, The relation $g \circ f : A \rightharpoonup C$ is a partial function.

PROOF: For $a \in A$ and $c_1, c_2 \in C$.

RTP: $a (g \circ f) c_1 \wedge a (g \circ f) c_2 \overset{?}{\Longrightarrow} c_1 = c_2$

Assume (1) $a (g \circ f) c_1$ and (2) $a (g \circ f) c_2$.

Then, by (1), $\exists b \in B.\ a f b \wedge b g c_1$.

By ②, $\exists\, b \in B \,.\, a f b \wedge b g c_2$

Let $b_1 \in B$ be such that

$$a f b_1 \wedge b_1 g c_1$$

and let $b_2 \in B$ be such that

$$a f b_2 \wedge b_2 g c_2 .$$

Then, as f is functional,

$$b_1 = b_2$$

Therefore,

$$b_1 g c_1 \text{ and } b_1 g c_2$$

Since, g is functional, $c_1 = c_2$ as required $\boxtimes$

In fact, for $a \in A$,

$$(g \circ f)(a) = \begin{cases} \uparrow & \text{, if } f(a)\uparrow \\ \uparrow & \text{, if } f(a)\downarrow \text{ but } g(f(a))\uparrow \\ g(f(a)) & \text{, if } f(a)\downarrow \text{ and } g(f(a))\downarrow \end{cases}$$

Theorem 121 *The identity relation is a partial function, and the composition of partial functions yields a partial function.*

NB

$$f = g : A \rightharpoonup B$$

iff

$$\forall a \in A. \big(f(a){\downarrow} \iff g(a){\downarrow} \big) \wedge f(a) = g(a)$$

Proposition 122 *For all finite sets* A *and* B*,*

$$\#(A \rightharpoonup B) \;=\; (\#B+1)^{\#A} \quad .$$

PROOF IDEA:

Say $A = \{a_1, a_2, \cdots, a_m\}$

$B = \{b_1, b_2, \cdots, b_n\}$

$$\begin{array}{ccccc}
a_1 & a_2 & \cdots & a_m & \\
\downarrow & \downarrow & & \downarrow & \\
\uparrow & \uparrow & & \uparrow & \\
b_1 \times & b_1 \times & \cdots \times & b_1 & \\
\vdots & \vdots & & \vdots & \\
b_n & b_n & & b_n &
\end{array}$$

$$(n+1) \times (n+1) \times \cdots \times (n+1)$$

$$\|$$

$$(n+1)^m$$