Preorders

Definition 116  A preorder $(\mathcal{P}, \sqsubseteq)$ consists of a set $\mathcal{P}$ and a relation $\sqsubseteq$ on $\mathcal{P}$ (i.e. $\sqsubseteq \in \mathcal{P}(\mathcal{P} \times \mathcal{P})$) satisfying the following two axioms.

- **Reflexivity.**
  \[
  \forall x \in \mathcal{P}. \ x \sqsubseteq x
  \]

- **Transitivity.**
  \[
  \forall x, y, z \in \mathcal{P}. \ (x \sqsubseteq y \land y \sqsubseteq z) \implies x \sqsubseteq z
  \]
Definition: A partial order \((P, \leq)\) is a preorder satisfying antisymmetry; that is,
\[\forall x, y \in P. (x \leq y \land y \leq x) \Rightarrow x = y.\]

Examples:

- \((\mathbb{R}, \leq)\) and \((\mathbb{R}, \geq)\).

- \((\mathcal{P}(A), \subseteq)\) and \((\mathcal{P}(A), \supseteq)\).

- \((\mathbb{Z}, |)\).

For all positive integers \(n\), we have \(n! / -n\) and \(-n! / n\) but \(n \neq -n\).
Theorem 118  For $R \subseteq A \times A$, let

$$\mathcal{F}_R = \{ Q \subseteq A \times A \mid R \subseteq Q \land Q \text{ is a preorder} \}.$$ 

Then, (i) $R^\circ \in \mathcal{F}_R$ and (ii) $R^\circ \subseteq \bigcap \mathcal{F}_R$. Hence, $R^\circ = \bigcap \mathcal{F}_R$.

**Proof:**

$$\bigcap \mathcal{F}_R \subseteq R^\circ \quad \text{and} \quad R^\circ \subseteq \bigcap \mathcal{F}_R.$$ 

$\bigcap \mathcal{F}_R \subseteq R^\circ$
(1) $R^* \subseteq F_R$

RTP (1) $R \subseteq R^*$

and (2) $R^*$ is a pre order.

(2) $R \subseteq R^*$

Let $x, y \in A$ be such that $x R y$.

It follows that there is a path from $x$ to $y$. Therefore, $x \ (R^*) y$. 
(2) \( R^{*} \) is a preorder.

(2.1) \( R^{*} \) is reflexive

If \( \forall x \in A \), \( x \ R^{*} x \)

Let \( x \) be in \( A \). Then, we have a path of length 0 from \( x \) to itself and, therefore, \( x \ (R^{*}) x \).

(2.2) \( R^{*} \) is transitive.
∀x, y, z ∈ A. x \mathcal{R}_0^* y \land y \mathcal{R}_0^* z \Rightarrow x \mathcal{R}_0^* z.

Assume x, y, z ∈ A such that x \mathcal{R}_0^* y and y \mathcal{R}_0^* z.

\text{RIP } x \mathcal{R}_0^* z.

By (1), there is a path from x to y in \mathcal{R}.

By (2), there is a path from y to z in \mathcal{R}.

Therefore, there is a path from x to z in \mathcal{R}.

It follows that x \mathcal{R}_0^* z as required.
(ii) $R^{o*} \subseteq \bigcap Fr$

If $\forall Q \in Fr. R^{o*} \subseteq Q$

Let $Q$ be a relation on $A$ that is a preorder and contains $R$.

RTP: $R^{o*} \subseteq Q$

$\bigcup \cap R_{on}$
$\cup_{\text{new}} R_{\text{on}} \subseteq Q$

$\forall \text{new}. R_{\text{on}} \subseteq Q$

Proof by induction.

**Base Case**: $RTP: R^{o(0)} \subseteq Q$

Since $Q$ is reflexive, $\forall x \in A. x \in Q$ and therefore $\text{id}_A \subseteq Q$. 
**Inductive Step:**

Assume

\[(IH)\]    \(R^n \subseteq Q\) for new

\[\text{RTP}\]

\[
\overline{R^n(\text{I}H)} \subseteq Q
\]

\[
\iff
\]

\[
R \circ R^n
\]

\[\forall x, y \in A. x (R \circ R^n) y \implies x \in Q \land y \in Q\]

Let \(x, y \in A\) such that \(x (R \circ R^n) y\).
\( x (R \circ R^o) y \)

\[\iff \exists z. xRz \land zR^oy.\]

By (IH), \( zQy.\) Moreover, \( xQz.\)

Therefore, by transitivity of \( Q, \) it follows that \( xQy \) as required.