Preorders

Definition 116 *A* preorder $(P, \sqsubseteq)$ *consists of a set* P *and a relation* $\sqsubseteq$ *on* P *(i.e.* $\sqsubseteq \in \mathcal{P}(P \times P)$*) satisfying the following two axioms.*

- *Reflexivity.*

$$\forall x \in P.\ x \sqsubseteq x$$

- *Transitivity.*

$$\forall x, y, z \in P.\ (x \sqsubseteq y \wedge y \sqsubseteq z) \implies x \sqsubseteq z$$

Def: A partial order $(P, \sqsubseteq)$ is a preorder satisfying ANTISYMMETRY; that is,

$$\forall x, y \in P.\ (x \sqsubseteq y \wedge y \sqsubseteq x) \Rightarrow x = y.$$

Examples:

- $(\mathbb{R}, \leq)$ and $(\mathbb{R}, \geq)$.
- $(\mathcal{P}(A), \subseteq)$ and $(\mathcal{P}(A), \supseteq)$.
- $(\mathbb{Z}, \mid)$.

For all positive integers n, we have $n \mid -n$ and $-n \mid n$ but $n \neq -n$.

Theorem 118 *For* $R \subseteq A \times A$, *let*

$$\mathcal{F}_R = \{ Q \subseteq A \times A \mid R \subseteq Q \wedge Q \textit{ is a preorder} \} \quad .$$

Then, $\underline{(i)\ R^{\circ *} \in \mathcal{F}_R \textit{ and}}$ $\underline{(ii)\ R^{\circ *} \subseteq \bigcap \mathcal{F}_R}$. *Hence,* $R^{\circ *} = \bigcap \mathcal{F}_R$.

PROOF:

$\Downarrow$

$\bigcap \mathcal{F}_R \subseteq R^{\circ *}$

(i) $R^{\circ *} \in \mathcal{F}_R$

RTP (1) $R \subseteq R^{\circ *}$

and (2) $R^{\circ *}$ is a preorder.

(1) $R \subseteq R^{\circ *}$

Nts $\forall x, y \in A.\ x R y \Rightarrow x (R^{\circ *}) y$

Let $x, y \in A$ be such that $x R y$.

It follows that there is a path from x to y. Therefore, $x (R^{\circ *}) y$.

(2) $R^{\circ *}$ is a preorder.

(2.1) $R^{\circ *}$ is reflexive

TS $\forall x \in A.\ x R^{\circ *} x$

Let x be in A. Then, we have a path of length 0 from x to itself and, therefore, $x (R^{\circ *}) x$.

(2.2) $R^{\circ *}$ is transitive

TS

$$\forall x, y, z \in A.\ x R^{\circ *} y \wedge y R^{\circ *} z \Longrightarrow x R^{\circ *} z.$$

Assume $x, y, z \in A$ such that (1) $x R^{\circ *} y$ and (2) $y R^{\circ *} z$.

<u>RTP</u> $x R^{\circ *} z$.

By (1), there is a path from x to y in R

By (2), there is a path from y to z in R

Therefore, there is a path from x to z in R.

It follows that $x R^{\circ *} z$ as required $\boxtimes$

(ii) $R^{\circ *} \subseteq \bigcap \mathcal{F}_R$

Iff

$$\forall Q \in \mathcal{F}_R .\ R^{\circ *} \subseteq Q$$

Let Q be a relation on A that is a preorder and contains R.

RTP: $R^{\circ *} \subseteq Q$

$$R^{\circ *} = \bigcup_{n \in \mathbb{N}} R^{\circ n}$$

$$\bigcup_{n \in \mathbb{N}} R^{\circ n} \subseteq Q$$

$\Updownarrow$

$$\forall n \in \mathbb{N}.\ R^{\circ n} \subseteq Q$$

Proof by induction.

BASE CASE: RTP: $R^{\circ(0)} \subseteq Q$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\ \ \|$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\ \ \mathrm{id}_A$

Since Q is reflexive, $\forall x \in A.\ x\,Q\,x$ and therefore $\mathrm{id}_A \subseteq Q$.

INDUCTIVE STEP:

Assume

(IH) $R^{\circ n} \subseteq Q$ for $n \in \mathbb{N}$

<u>RTP</u> $R^{\circ(n+1)} \subseteq Q$

$$R^{\circ(n+1)} = R \circ R^{\circ n}$$

$\forall x, y \in A .\ x (R \circ R^{\circ n}) y \overset{?}{\Rightarrow} x Q y$

Let $x, y \in A$ such that $x (R \circ R^{\circ n}) y$.

$x \, (R \circ R^{\circ n}) \, y$

$\iff \exists z . \ \underline{x R z} \wedge \underline{z R^{\circ n} y}$

By (IH), $z Q y$. Moreover, $x Q z$. Therefore, by transitivity of Q, it follows that $x Q y$ as required.

$\boxtimes$