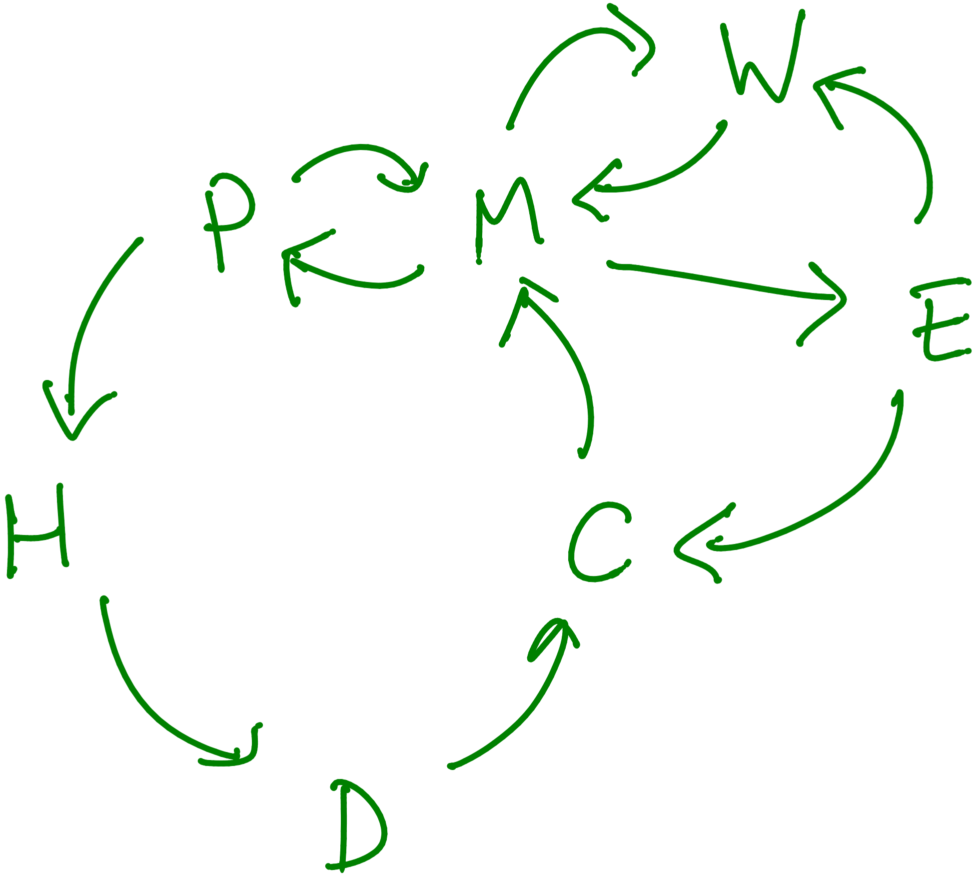
Directed graphs

Definition 108 *A* directed graph (A, R) *consists of a set* A *and a relation* R *on* A *(i.e. a relation from* A *to* A*).*



A = set of cities

$R \subseteq A \times A$

with aRb

iff there is a connection from a to b.

NB: Relations on a set may be composed with themselves.

For $R \subseteq A \times A$,

$R \circ R \subseteq A \times A$

For $a, b \in A$: $a (R \circ R) b \iff \exists c \in A.$
$aRc \wedge cRb$

That is,

$$\begin{tikzcd} a \arrow[r, bend left] & c \arrow[r, bend left] & b \end{tikzcd}$$

There is a path of length 2 from a to b.

$R^{\circ 3} := R \circ R \circ R \subseteq A \times A$

For $a, b \in A$;

$a \, (R^{\circ 3}) \, b \iff \exists c, d \in A.$

$aRc \wedge cRd \wedge dRb$

That is,

$$a \curvearrowright c \curvearrowright d \curvearrowright b$$

There a path from a to b of length 3.

Recall

$(\underline{Rel}(A), id_A, \circ)$ is a monoid.

$R^{\circ 1} := R$

$R^{\circ(n+1)} := R \circ R^{\circ(n)}$

$R^{\circ(0)} := id_A$

Paths

Proposition 113 *Let* (A, R) *be a directed graph. For all* $n \in \mathbb{N}$ *and* $s, t \in A$, $s \; R^{\circ n} \; t$ *iff there exists a path of length* n *in* R *with source* s *and target* t.

PROOF:

A path of length n (for $n \in \mathbb{N}$) is a sequence $\langle a_0, a_1, \ldots, a_{n-1}, a_n \rangle$ such that

$$a_0 \, R \, a_1, \; a_1 \, R \, a_2, \; \ldots, \; a_{n-1} \, R \, a_n.$$

NB

(1) The sequence $\langle a \rangle$ is a path of length 0 from a to itself.

(2) The sequence $\langle a, b \rangle$ is a path of length 1 from a to b iff the pair (a, b) is in the relation.

We show, for all $n \in \mathbb{N}$,

$s \, R^{\circ n} \, t$

$\Leftrightarrow$ There exists a path of length n from s to t in R.

PROOF: We proceed by induction.

BASE CASE:

$$s \, R^{\circ(0)} \, t$$

$\overset{?}{\iff}$ There exists a path of length 0 from s to t.

By definition, $R^{\circ(0)} = \mathrm{id}_A$. Therefore,

$$s \, R^{\circ(0)} \, t \iff s = t.$$

On the other hand,

$\vdots$

there is a path of length 0 from s to t

$\iff$ $s = t$.

That is, the BASE CASE holds.

INDUCTIVE STEP: For $n \in \mathbb{N}$,

Assume

$$s \, (R^{\circ n}) \, t$$

(IH) $\iff$ there exists a path of length n from s to t in R.

RTP: $s \, R^{\circ(n+1)} \, t$

$\overset{?}{\iff}$ There exists a path of length n+1 from s to t in R.

($\Rightarrow$) Assume $s \, R^{\circ(n+1)} \, t$; equivalently

$s \, (R \circ R^{\circ n}) \, t$

$\iff$

$\exists x \in A . \; s R x \wedge \underline{x \, R^{\circ(n)} \, t}$

By (IH), There exist a path of length n from x to t; That is, a sequence

$$\langle \underset{\substack{\| \\ x}}{a_0}, a_1, \dots, \underset{\substack{\| \\ t}}{a_n} \rangle$$

and

$$a_0 R a_1 R \cdots a_{n-1} R a_n .$$

Then,

$$\langle s, a_0, a_1, \dots, \underset{\substack{\| \\ t}}{a_n} \rangle$$

Since $s R a_0$ we have a path of length

$n+1$ from s to t as required.

($\Leftarrow$) If there is a path of length $n+1$ from s to t then $s \; R^{o(n+1)} \; t$.

Assume

$$\langle \underset{\underset{s}{\parallel}}{a_0}, a_1, \ldots, a_n, \underset{\underset{t}{\parallel}}{a_{n+1}} \rangle$$

such that $\underset{\underset{s}{\parallel}}{a_0} R a_1 R \ldots a_n R \underset{\underset{t}{\parallel}}{a_{n+1}}$

Then we have a path of length n from a_1 to t; namely,

$$\langle a_1, a_2, \cdots, a_n, \underset{\substack{\shortparallel \\ t}}{a_{n+1}} \rangle$$

By (IH), $a_1 R^{\circ n} t$ and since $s R a_1$ we have $s \underbrace{\left(R \circ R^{\circ(n)}\right)}_{\substack{\shortparallel \\ R^{\circ(n+1)}}} t$ and we are done.

$\boxtimes$

NB: $R^{\circ *} = \{ (s,t) \in A \times A \mid \exists n \in \mathbb{N} .\ s \, R^{\circ n} \, t \}$

Definition 114 *For* $R \in \mathrm{Rel}(A)$, *let*

$$R^{\circ *} \;=\; \bigcup \{ R^{\circ n} \in \mathrm{Rel}(A) \mid n \in \mathbb{N} \} \;=\; \bigcup_{n \in \mathbb{N}} R^{\circ n} \quad .$$

Corollary 115 *Let* (A, R) *be a directed graph. For all* $s, t \in A$, $s \; R^{\circ *} \; t$ *iff there exists a path with source* s *and target* t *in* R.

Application:

Let $R \subseteq [n] \times [n]$ $\quad [n] = \{0, 1, \dots, n-1\}$

Recall that R can equivalently represented by an $(n \times n)$-matrix

$$M = \underline{mat}(R)$$

The adjenceey matrix of R

We then have:

$$\underline{mat}(R^{\circ m}) = \overbrace{M \cdot \dots \cdot M}^{m \text{ times}} = M^{m}$$

Define

$$M^0 = I$$

$$M^{m+1} = M \cdot M^m \quad (m \in \mathbb{N})$$

Moreover, for $m \in \mathbb{N}$,

$$\underline{mat}\Big(\bigcup_{0 \le i \le m} R^{\circ i}\Big) = \sum_{0 \le i \le m} \underline{mat}(R^{\circ i})$$

$$= \sum_{0 \le i \le m} M^i$$

Define

$$M_0 = I$$

$$M_{m+1} = I + M \cdot M_m$$

Then

$$\underline{\mathrm{mat}}\left(\bigcup_{0 \le i \le m} R^{\circ i} \right) = M_m$$

Finally, as R is a relation on an n-element set,

$$R^{\circ *} = \bigcup_{0 \le i \le n} R^{\circ i}$$

Therefore

$$R^{o*}$$

can be calculated as

$$\underline{\text{rel}}(M_n).$$

The $(n \times n)$-matrix $M = \mathrm{mat}(R)$ of a finite directed graph $\big([n], R\big)$ for n a positive integer is called its ***adjacency matrix***.

The adjacency matrix $M^* = \mathrm{mat}(R^{\circ *})$ can be computed by matrix multiplication and addition as M_n where

$$\begin{cases} M_0 &= I_n \\ M_{k+1} &= I_n + \big(M \cdot M_k\big) \end{cases}$$

This gives an algorithm for establishing or refuting the existence of paths in finite directed graphs.