**Definition 103**

1. *For positive integers $m$ and $n$, an $(m \times n)$-matrix $M$ over a semiring $(S, 0, \oplus, 1, \odot)$ is given by entries $M_{i,j} \in S$ for all $0 \leq i < m$ and $0 \leq j < n$.*

**Theorem 104** *Matrix multiplication is associative and has the identity matrix as neutral element.*

# MATRIX MULTIPLICATION

For an $(m \times n)$-matrix

$$M = (M_{i,j})_{0 \leq i < m,\, 0 \leq j < n}$$

and an $(n \times \ell)$-matrix

$$N = (N_{i,j})_{0 \leq i < n,\, 0 \leq j < \ell}$$

we define

$(N \cdot M)$ $(m \times \ell)$-matrix

$$(N \cdot M)_{i,j} \stackrel{def}{=} \sum_{0 \leq k < n} M_{i,k} \cdot N_{k,j} .$$

# Identity Matrix

$I$ $(m \times m)$ - matrix

$$I_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otw} \end{cases}$$

## LAWS

$N \cdot I = N$ , $I \cdot M = M$ , $L \cdot (M \cdot N) = (L \cdot M) \cdot N$

# BOOLEAN SEMIRING

$\mathrm{Bool} = \{\mathrm{true}, \mathrm{false}\}$

$(\mathrm{false}, \vee)$ additive structure

$(\mathrm{true}, \wedge)$ multiplicative structure

**NB**

$$(N \cdot M)_{i,j} = \bigvee_{k} M_{i,k} \wedge N_{k,j}$$

Recall: $[k] = \{0, 1, \ldots, k-1\}$

Relations from $[m]$ to $[n]$ and $(m \times n)$-matrices over Booleans provide two alternative views of the same structure.

This carries over to identities and to composition/multiplication .

- Given an $(m \times n)$-matrix $M$

Define $\underline{rel}(M) : [m] \rightarrow\!\!\!\!+ [n]$

$$\overset{\text{def}}{=} \{ (i,j) \in [m] \times [n] \mid M_{i,j} = \text{true} \}$$

(3×2)-matrix

$$M = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & \text{true} & \text{false} \\ 1 & \text{false} & \text{true} \\ 2 & \text{false} & \text{false} \end{array}$$

$$\underline{\text{rel}}(M) : [3] \nrightarrow [2]$$

$$\underline{\text{rel}}(M) = \{ (0,0), (1,1) \}$$

- Given $R : [m] \to\!\!\!\!\!+ [n]$

Define the $(m \times n)$-matrix $\underline{mat}(R)$

$$\underline{mat}(R)_{i,j} = \begin{cases} \text{true} & \text{if } (i,j) \in R \\ \text{false} & \text{o/w.} \end{cases}$$

**Proposition:**

$$\underline{rel}(\underline{mat}(R)) = R$$

$$\underline{mat}(\underline{rel}(M)) = M$$

**Example:** $R : [3] \leftrightarrow [2]$

$$= \{(0,0), (0,1), (1,1), (2,1)\}$$

$$\text{mat}(R) = \begin{array}{c} 0 \\ 1 \\ 2 \end{array}\begin{bmatrix} \text{true} & \text{true} \\ \text{false} & \text{true} \\ \text{false} & \text{true} \end{bmatrix}$$

$$\qquad\qquad\qquad\quad 0 \qquad\quad 1$$

$$\text{rel}(\text{mat}(R)) = \{(0,0), (0,1), (1,1), (2,1)\} = R$$

PROOF: $\underline{rel}(\underline{mat}(R)) = R$

RTP: For all $i, j$:

$$(i, j) \in \underline{rel}(\underline{mat}(R)) \overset{?}{\iff} (i, j) \in R$$

$$(i, j) \in \underline{rel}(\underline{mat}(R))$$

$$\iff \underline{mat}(R)\ i, j = \text{true}$$

$$\iff (i, j) \in R$$

$\boxtimes$

PROOF: $\underline{mat}(\underline{rel}(M)) = M$

RTP: For all $i, j$.

$$\underline{mat}(\underline{rel}(M))_{i,j} = M_{i,j}$$

$$\underline{mat}(\underline{rel}(M))_{i,j} = \begin{cases} \text{true} & \text{if } (i,j) \in \underline{rel}(M) \\ \text{false} & \text{otw} \end{cases}$$

$$= \begin{cases} \text{true} & \text{if } M_{i,j} = \text{true} \\ \text{false} & \text{otw} \end{cases}$$

$$= M_{i,j}$$

☒

**Proposition:** For $R : [m] \nrightarrow [n]$ and $S : [n] \nrightarrow [\ell]$,

$$\underline{\text{mat}}(S \circ R) = \underline{\text{mat}}(S) \cdot \underline{\text{mat}}(R).$$

**PROOF:** $R : [m] \nrightarrow [n]$, $S : [n] \nrightarrow [\ell]$.
Then, for $i \in [m]$ and $j \in [n]$, we have:

$$\left(\underline{\text{mat}}(S) \cdot \underline{\text{mat}}(R)\right)_{i,j}$$

$$= \bigvee_k \underline{\text{mat}}(R)_{i,k} \wedge \underline{\text{mat}}(S)_{k,j}$$

$$= \bigvee_k (i,k) \in R \wedge (k,j) \in S$$

Therefore:

$$(\underline{mat}(S) \cdot \underline{mat}(R))_{i,j} = \text{true}$$

$$\iff \exists k \in [n].\ (i,k) \in R \land (k,j) \in S$$

$$\iff (i,j) \in (S \circ R)$$

$$\iff \underline{mat}(S \circ R)_{i,j} = \text{true}.$$

$\boxtimes$

**Corollary:** For relations $R : [m] \nrightarrow [n]$ and $S : [n] \nrightarrow [\ell]$,

$$S \circ R = \underline{\text{rel}}\left( \underline{\text{mat}}(S) \cdot \underline{\text{mat}}(R) \right)$$

**Remark:**

$$\underline{\text{rel}}(I) = \text{id} \quad \text{and} \quad \underline{\text{mat}}(\text{id}) = I$$

# MATRIX ADDITION

For $(m \times n)$-matrices $M$ and $N$, $M+N$ is the $(m \times n)$-matrix with entries

$$(M+N)_{i,j} = M_{i,j} + N_{i,j}$$

It is an associative operation with neutral element the $(m \times n)$-matrix $Z$ with entries

$$Z_{i,j} = 0$$

**Proposition** For relations $R, S : [m] \nrightarrow [n]$,

$$\underline{mat}(R) + \underline{mat}(S) = \underline{mat}(R \cup S)$$

and

$$Z = \underline{mat}(\emptyset)$$

<u>NB</u> $(m\times m)$-matrices with additive structure $Z, +$ and multiplicative structure $I, \cdot$ form a semiring