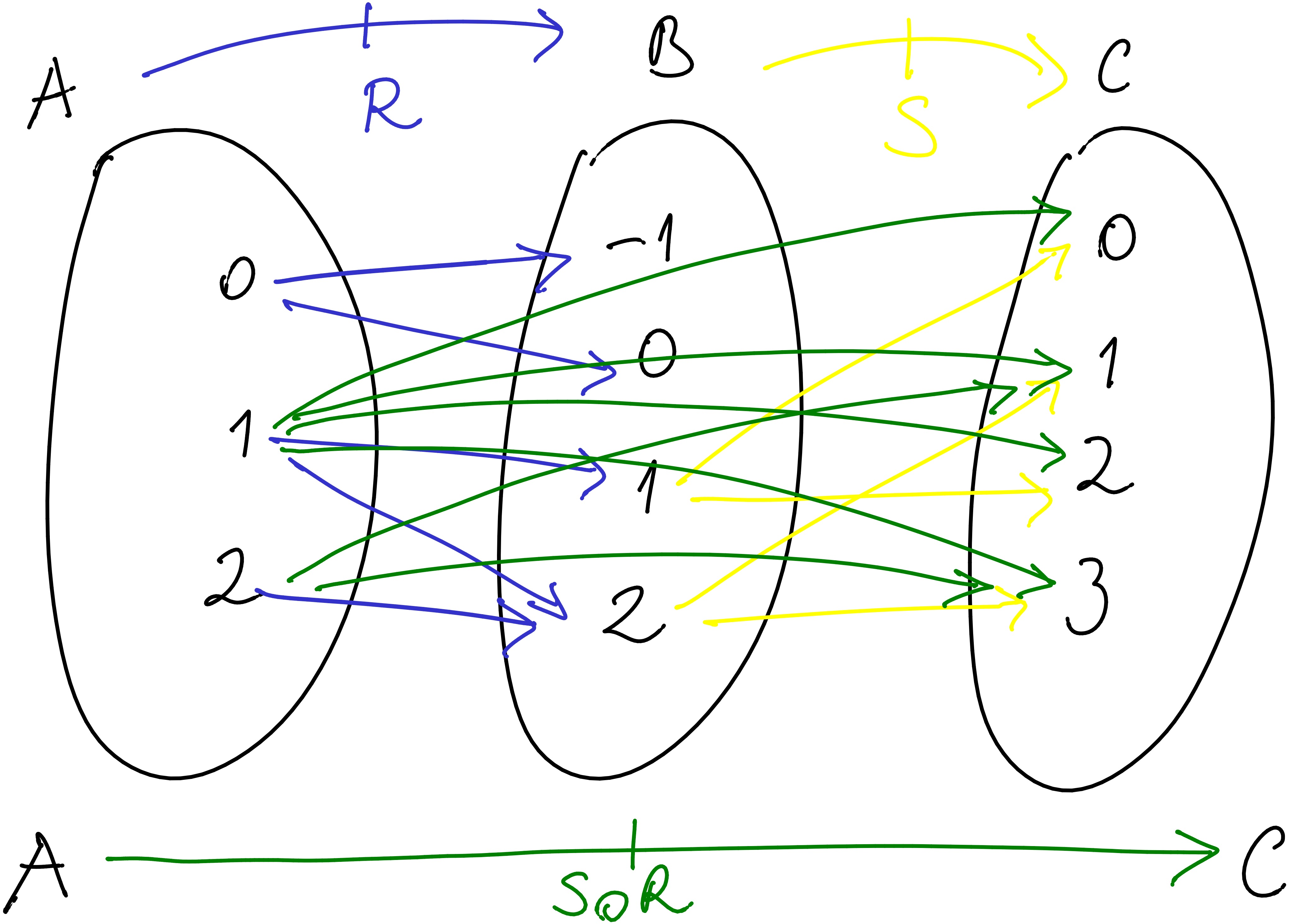
Relational composition

$$\frac{A \overset{R}{\nrightarrow} B \qquad B \overset{S}{\nrightarrow} C}{A \overset{S \circ R}{\nrightarrow} C}$$


A
R
B
S
C
0
1
2
-1
0
1
2
0
1
2
3
A
S∘R
C

RELATIONAL COMPOSITION

Given $R : A \nrightarrow B$ and $S : B \nrightarrow C$,

Define $S \circ R : A \nrightarrow C$

$$\overset{\text{def}}{=} \{ (a,c) \in A \times C \mid \exists b \in B .\ aRb \wedge bSc \}$$

In other words:

$$a (S \circ R) c \overset{\text{def}}{\iff} \exists b \in B .\ aRb \wedge bSc .$$

Examples

(1) $Sq : \mathbb{R}_{\geq 0} \not\to \mathbb{R}$ $\qquad a \, Sq \, b \overset{\text{def}}{\iff} a = b^2$

$Neg : \mathbb{R} \not\to \mathbb{R}$ $\qquad x \, Neg \, y \overset{\text{def}}{\iff} x = -y$

Claim: $Neg \circ Sq = Sq : \mathbb{R}_{\geq 0} \not\to \mathbb{R}$

RTP For $p \in \mathbb{R}_{\geq 0}$ and $q \in \mathbb{R}$,

$$p \, (Neg \circ Sq) \, q \overset{?}{\iff} p \, Sq \, q$$

$$p\,(\mathrm{Neg}\circ \mathrm{Sq})\,q$$

$$\iff \exists r \in \mathbb{R}.\ p\ \mathrm{Sq}\ r \wedge r\ \mathrm{Neg}\ q$$

$$\iff \exists r \in \mathbb{R}.\ p = r^2 \wedge r = -q$$

$$\iff p = (-q)^2$$

$$\iff p = q^2$$

$$\iff p\ \mathrm{Sq}\ q. \qquad \boxtimes$$

(2) For $R : A \nrightarrow B$ and $S : B \nrightarrow C$

define

$$S \bowtie R \subseteq A \times B \times C$$

$$\overset{\text{def}}{=}$$

$$\{(a,b,c) \in A \times B \times C \mid (a,b) \in R \wedge (b,c) \in S\}$$

For $T \subseteq A \times B \times C$

define

$$P_{1,3}(T) : A \nrightarrow C$$

$$\overset{\text{def}}{=}$$

$$\{(a,c) \in A \times C \mid \exists b \in B.\, (a,b,c) \in T\}$$

Claim: $P_{1,3}(S \bowtie R) = S \circ R$

<u>RTP</u>: For all $a \in A$, $c \in C$,

$$(a,c) \in \pi_{1,3}(S \bowtie R)$$

iff

$$(a,c) \in S \circ R$$

$\boxtimes$

NB: $(\underline{\mathrm{Rel}}(A,A), \mathrm{id}_A, \circ)$ is a monoid.

Theorem 102 *Relational composition is associative and has the identity relation as neutral element.*

- *Associativity.*

  For all $R : A \rightarrow\!\!\!\!\!+ B$, $S : B \rightarrow\!\!\!\!\!+ C$, *and* $T : C \rightarrow\!\!\!\!\!+ D$,

$$(T \circ S) \circ R \;=\; T \circ (S \circ R)$$

- *Neutral element.*

  For all $R : A \rightarrow\!\!\!\!\!+ B$,

$$R \circ \mathrm{id}_A \;=\; R \;=\; \mathrm{id}_B \circ R \quad .$$

Proposition: For $R : A \nrightarrow B$, $S : B \nrightarrow C$, and $T : C \nrightarrow D$,

$$(T \circ S) \circ R = T \circ (S \circ R) : A \nrightarrow D.$$

PROOF:

RTP For $a \in A$, $d \in D$,

$$a \, ((T \circ S) \circ R) \, d \overset{?}{\iff} a \, (T \circ (S \circ R)) \, d$$

$a \, ((T \circ S) \circ R) \, d$

$\iff \exists \, b \in B. \; a R b \wedge b \, (T \circ S) \, d$

$\iff \exists \, b \in B. \; a R b \wedge \exists \, c \in C. \; b S c \wedge c T d$

$\iff \exists b \in B.\ \exists c \in C.\ aRb \wedge bSc \wedge cTd$ (1)

$a\,(T \circ (S \circ R))\,d$

$\iff \exists c \in C.\ a(S \circ R)c \wedge cTd$

$\iff \exists c \in C.\ \exists b \in B.\ aRb \wedge bSc \wedge cTd$ (2)

Since (1) $\iff$ (2) we are done. $\boxtimes$