

Relations

Definition 99 A (binary) relation R from a set A to a set B

$$R : A \dashrightarrow B \quad \text{or} \quad R \in \text{Rel}(A, B) \quad ,$$

is

$$R \subseteq A \times B \quad \text{or} \quad R \in \mathcal{P}(A \times B) \quad .$$

Notation 100 One typically writes $a R b$ for $(a, b) \in R$.

$$R \subseteq A \times B$$

$$a \in A, b \in B$$

$$a R b$$

$$\Uparrow (a, b) \in R$$

PROGRAM SEMANTICS

$$\underline{sq} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$$

given by all pairs

$$(x, y) \text{ such that } x = y^2$$

In particular, $1 \underline{sq} 1$

$$1 \underline{sq} -1$$

TYPING

$P : \alpha$

E.g.

$(\underline{fn} \ x \rightarrow x, \ \underline{bool} \rightarrow bool)$

$(\underline{fn} \ x \rightarrow x, \ \underline{nat} \rightarrow nat)$

are in the typing relation

$(\underline{fn} \ x \rightarrow x, \ \underline{bool} \rightarrow \underline{nat})$

is not.

NETWORKS

N — nodes

C — connections

$$C: N \rightarrow N$$

DATABASES

A relation R on sets A_1, A_2, \dots, A_n is defined as a subset

$$R \subseteq A_1 \times A_2 \times \dots \times A_n$$

E.g. $R \subseteq \text{Movies} \times \text{Directors} \times \text{Years} \times \text{Person}$
consisting of all quadruples (m, d, y, p)
such that movie m was directed by director
 d in year y with person p a cast member.

Examples:

- ▶ Empty relation.

$$\emptyset : A \dashrightarrow B$$

$$(a \emptyset b \iff \text{false})$$

- ▶ Full relation.

$$(A \times B) : A \dashrightarrow B$$

$$(a (A \times B) b \iff \text{true})$$

- ▶ Identity (or equality) relation.

$$\text{id}_A = \{ (a, a) \mid a \in A \} : A \dashrightarrow A$$

$$(a \text{id}_A a' \iff a = a')$$

- ▶ Integer square root.

$$R_2 = \{ (m, n) \mid m = n^2 \} : \mathbb{N} \dashrightarrow \mathbb{Z}$$

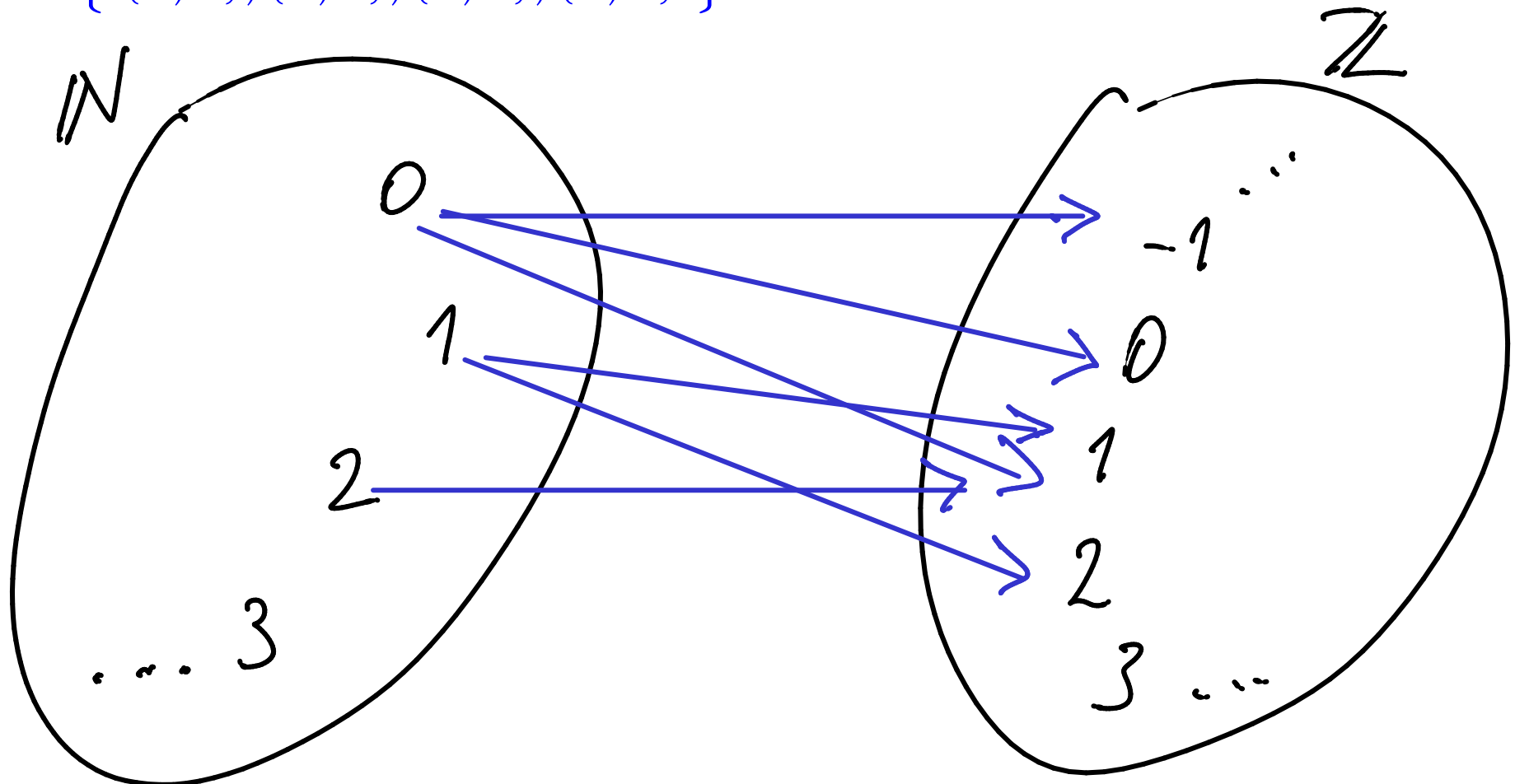
$$(m R_2 n \iff m = n^2)$$

Internal diagrams

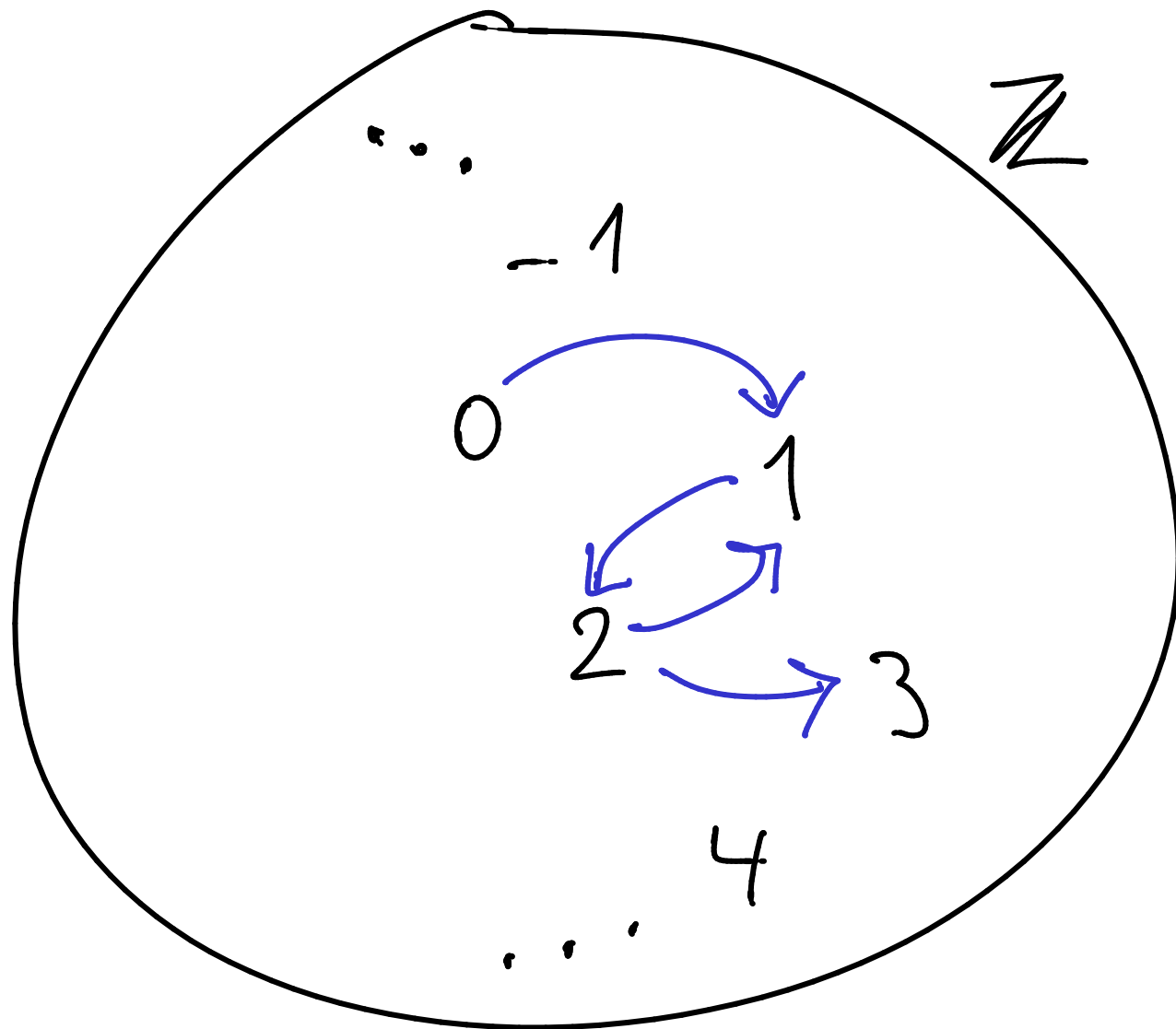
Example:

$$R = \{ (0, 0), (0, -1), (0, 1), (1, 2), (1, 1), (2, 1) \} : \mathbb{N} \dashrightarrow \mathbb{Z}$$

$$S = \{ (1, 0), (1, 2), (2, 1), (2, 3) \} : \mathbb{Z} \dashrightarrow \mathbb{Z}$$



$$S = \{(0,1), (1,2), (2,1), (2,3)\}$$



Relational extensionality

$$R = S : A \rightarrow B$$

iff

$$\forall a \in A. \forall b \in B. a R b \iff a S b$$