BIG

UNIONS and INTERSECTIONS

Sets and logic

$\mathcal{P}(\mathcal{U})$	$\{\,\textbf{false}\,,\ \textbf{true}\,\}$
$\emptyset$	**false**
$\mathcal{U}$	**true**
$\cup$	$\vee$
$\cap$	$\wedge$
$(\cdot)^{c}$	$\neg(\cdot)$
$\bigcup$	$\exists$
$\bigcap$	$\forall$

Example: Big union

- $\mathcal{T} =^{def} \{ T \subseteq [5] \mid$ the sum of the elements of T is less than or equal 2 $\}$

$$= \{ \emptyset, \{0\}, \{1\}, \{0,1\}, \{0,2\} \}$$

- $\bigcup \mathcal{T}$ is the union of the sets in $\mathcal{T}$

$$n \in \bigcup \mathcal{T} \iff \exists T \in \mathcal{T}.\, n \in T$$

$$\bigcup \mathcal{T} = \{0, 1, 2\}$$

Big unions

Definition 90 *Let $\mathcal{U}$ be a set. For a collection of sets $\mathcal{F} \in \mathcal{P}(\mathcal{P}(\mathcal{U}))$, we let the* big union *(relative to $\mathcal{U}$) be defined as*

$$\bigcup \mathcal{F} = \{ x \in \mathcal{U} \mid \exists A \in \mathcal{F}. x \in A \} \in \mathcal{P}(\mathcal{U}) \quad .$$

ASSOCIATIVITY (idea / intuition)

$$\mathcal{F} \subseteq \mathcal{P}(\mathcal{P}(U))$$

$$\mathcal{F} = \{ \dots, \underset{\{\dots, A, A', \dots\}}{\underset{\parallel}{\mathcal{A}}}, \overset{\{\dots, B, B', \dots\}}{\overset{\parallel}{\mathcal{B}}}, \dots \}$$

- $\bigcup \mathcal{F} = \dots \cup \mathcal{A} \cup \mathcal{B} \cup \dots$
$$= \{ \dots, \dots A, A', \dots, \dots B, B', \dots, \dots \}$$
$$\bigcup(\bigcup \mathcal{F}) = (\dots \dots \cup A \cup A' \cup \dots \dots \cup B \cup B' \cup \dots \dots)$$

- $\bigcup \{ \dots, \bigcup \mathcal{A}, \bigcup \mathcal{B}, \dots \}$
$$= \dots \cup (\dots \cup A \cup A' \cup \dots) \cup (\dots \cup B \cup B' \cup \dots) \cup \dots$$

Proposition 91 *For all* $\mathcal{F} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{U})))$,

$$\bigcup\left(\bigcup \mathcal{F}\right) = \bigcup\underbrace{\left\{ \bigcup \mathcal{A} \in \mathcal{P}(\mathbb{U}) \;\middle|\; \mathcal{A} \in \mathcal{F} \right\}} \in \mathcal{P}(\mathbb{U}) \quad .$$

PROOF:

NB(1): pattern-matching notation for

$$\{ X \in \mathcal{P}(\mathbb{U}) \mid \exists \mathcal{A} \in \mathcal{F}.\ X = \bigcup \mathcal{A} \}$$

NB(2): (Type-checking) as $\mathcal{F} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{U})))$ we have $\bigcup \mathcal{F} \in \mathcal{P}(\mathcal{P}(\mathbb{U}))$ and then $\bigcup(\bigcup \mathcal{F}) \in \mathcal{P}(\mathbb{U})$

PROOF: For $x \in U$, we show:

$$x \in \bigcup\left(\bigcup \mathcal{F}\right) \iff x \in \bigcup\{X \in \mathcal{P}(U) \mid \exists \mathcal{A} \in \mathcal{F}.\, X = \bigcup \mathcal{A}\}$$

On the one hand,

$$x \in \bigcup\left(\bigcup \mathcal{F}\right) \iff \exists S \in \bigcup \mathcal{F}.\ x \in S$$

$$\iff \exists \mathcal{A} \in \mathcal{F}.\ \exists S \in \mathcal{A}.\ x \in S$$

On the other hand,

$$x \in \bigcup\{X \in \mathcal{P}(U) \mid \exists \mathcal{A} \in \mathcal{F}.\ X = \bigcup \mathcal{A}\}$$

$$\iff \exists\ X \in \mathcal{P}(U).\ \exists \mathcal{A} \in \mathcal{F}.\ X = \bigcup \mathcal{A} \wedge x \in X$$

$$\iff \exists \mathcal{A} \in \mathcal{F}.\ x \in \bigcup \mathcal{A}$$

$$\iff \exists \mathcal{A} \in \mathcal{F}.\ \exists S \in \mathcal{A}.\ x \in S.$$

$\boxtimes$

Examples:

- $\bigcup(\mathcal{P}(U)) = \{x \in U \mid \exists S \in \mathcal{P}(U).\ x \in S\}$

  $= \{x \in U \mid \underline{\text{true}}\} = U$

- $\bigcup \emptyset = \{x \in U \mid \exists S \in \emptyset.\ x \in S\}$

  $= \{x \in U \mid \underline{\text{false}}\} = \emptyset$

Example: Big intersection

- $\mathcal{S} \stackrel{\text{def}}{=} \{ S \subseteq [5] \mid$ the sum of the elements of S equals $6 \}$

  $= \{ \{2,4\}, \{0,2,4\}, \{1,2,3\} \}$

- $\bigcap \mathcal{S}$ is the intersection of the sets in $\mathcal{S}$

  $n \in \bigcap \mathcal{S} \iff \forall S \in \mathcal{S}.\ n \in S$

  $\bigcap \mathcal{S} = \{2\}$

Big intersections

Definition 92 *Let* $\mathcal{U}$ *be a set. For a collection of sets* $\mathcal{F} \subseteq \mathcal{P}(\mathcal{U})$, *we let the* big intersection *(relative to* $\mathcal{U}$*) be defined as*

$$\bigcap \mathcal{F} = \{ x \in \mathcal{U} \mid \forall A \in \mathcal{F}. x \in A \} \quad .$$

Examples:

- $$\bigcap(\mathcal{P}(U)) = \{x \in U \mid \forall S \in \mathcal{P}(U).\, x \in S\}$$
  $$= \{x \in U \mid \underline{\text{false}}\} = \emptyset$$

- $$\bigcap \emptyset = \{x \in U \mid \forall S \in \emptyset.\, x \in S\}$$
  $$= \{x \in U \mid \text{true}\} = U.$$

Theorem 93 *Let*

$$\mathcal{F} = \left\{ S \subseteq \mathbb{R} \;\middle|\; (0 \in S) \wedge (\forall x \in \mathbb{R}.\, x \in S \implies (x+1) \in S) \right\} .$$

Then, (i) $\mathbb{N} \in \mathcal{F}$ *and* (ii) $\mathbb{N} \subseteq \bigcap \mathcal{F}$. *Hence,* $\bigcap \mathcal{F} = \mathbb{N}$.

PROOF: Clearly $\mathbb{N} \in \mathcal{F}$. So we prove (ii)

RTP $\forall n \in \mathbb{N}.\, n \in \bigcap \mathcal{F}$

We proceed by induction.

BASE CASE $\forall S \in \mathcal{F}.\, 0 \in S$

For $S \in \mathcal{F}$, we have $S \subseteq \mathbb{R}$ with $0 \in S$.

INDUCTIVE STEP:

Let $n \in \mathbb{N}$ (1) and assume

(IH) $\forall S \in \mathcal{F}.\ n \in S.$

RTP: $\forall T \in \mathcal{F}.\ (n+1) \in T$

Let $T \in \mathcal{F}$.

Then, by (IH), we have $n \in T$ (2)

Also $\forall x \in \mathbb{R}.\ x \in T \Rightarrow (x+1) \in T$

By (1), $n \in T \Rightarrow (n+1) \in T$ (3)

By (2) and (3), $(n+1) \in T$.

$\boxtimes$

Proposition: Let U be a set and let $\mathcal{F} \subseteq \mathcal{P}(U)$ be a collection of subsets of U.

(1) For all $S \in \mathcal{P}(U)$,

$$S = \bigcup \mathcal{F}$$

iff

$$[\forall A \in \mathcal{F}.\ A \subseteq S]$$

and

$$[\forall X \in \mathcal{P}(U).\ (\forall A \in \mathcal{F}.\ A \subseteq X) \Rightarrow S \subseteq X]$$

(2) For all $T \in \mathcal{P}(U)$,

$$T = \bigcap \mathcal{F}$$

iff

$$[\forall A \in \mathcal{F}.\ T \subseteq A]$$

and

$$[\forall Y \in \mathcal{P}(U).\ (\forall A \in \mathcal{F}.\ Y \subseteq A) \Rightarrow Y \subseteq T]$$