Unordered & Ordered Pairing

Pairing axiom

For every a and b, there is a set with a and b as its only elements.

$$\{ a, b \}$$

defined by

$$\forall x.\, x \in \{a, b\} \iff (x = a \lor x = b)$$

NB The set $\{a, a\}$ is abbreviated as $\{ a \}$, and referred to as a *singleton*.

Proposition: For all a, b, c, x, y,

(1) $\{a\} = \{x, y\} \Rightarrow x = y = a$

(2) $\{c, x\} = \{c, y\} \Rightarrow x = y$

PROOF: Let a, b, c, x, y be arbitrary.

(1) Assume $\{a\} = \{x, y\}$

RTP: $x = y = a$.

Then $x \in \{a\}$ and therefore $x = a$.

Analogously, $y = a$.

(2) <u>Assume</u> $\{c, x\} = \{c, y\}$

<u>RTP</u>: $x = y$.

Then $[x = c \vee x = y]$

$\wedge [y = c \vee y = x]$

Therefore $(x = y = c) \vee (x = y)$

In either case

$x = y$

$\boxtimes$

ORDERED PAIRING

Notation:

$$(a,b) \quad \text{or} \quad \langle a,b\rangle$$

Fundamental property:

$$(a,b)=(x,y) \iff (a=x \wedge b=y)$$

Ordered pairing

For every pair a and b, the set

$$\{\{a\},\{a,b\}\}$$

is abbreviated as

$$\langle a,b\rangle$$

and referred to as an *ordered pair*.

Proposition 87 (Fundamental property of ordered pairing)
For all a, b, x, y,

$$\langle a, b \rangle = \langle x, y \rangle \iff (a = x \wedge b = y) \quad .$$

PROOF: Let a, b, x, y be arbitrary.

($\Leftarrow$) Vacuously.

($\Rightarrow$) Assume $\{\{a\}, \{a, b\}\} = \{\{x\}, \{x, y\}\}$.

RTP $a = x \wedge b = y$.

Then, $\{a\} = \{x\} \vee \{a\} = \{x, y\}$.

In either case, $a = x$.

Then

$$\{\{a\},\{a,b\}\} = \{\{a\},\{a,y\}\}.$$

$$\Rightarrow \quad \{a,b\} = \{a,y\}$$

$$\Rightarrow \quad b = y.$$

$\boxtimes$

Products

The *product* $A \times B$ of two sets A and B is the set

$$A \times B = \{ x \mid \exists\, a \in A, b \in B.\, x = (a, b) \}$$

where

$$\begin{aligned} &\forall a_1, a_2 \in A, b_1, b_2 \in B. \\ &\quad (a_1, b_1) = (a_2, b_2) \iff (a_1 = a_2 \wedge b_1 = b_2) \end{aligned} \quad .$$

Thus,

$$\forall\, x \in A \times B.\, \exists!\, a \in A.\, \exists!\, b \in B.\, x = (a, b) \quad .$$

PATTERN-MATCHING NOTATION

Example: The subset of ordered pairs from a set A with equal components is formally

$$\{ x \in A \times A \mid \exists a_1 \in A . \exists a_2 \in A . x = (a_1, a_2) \wedge a_1 = a_2 \}$$

but often abbreviated using pattern-matching notation as

$$\{ (a_1, a_2) \in A \times A \mid a_1 = a_2 \} .$$

Notation: For a property $P(a,b)$ with a ranging over a set A and b rangin over a set B,

$$\{(a,b) \in A \times B \mid P(a,b)\}$$

abbreviates

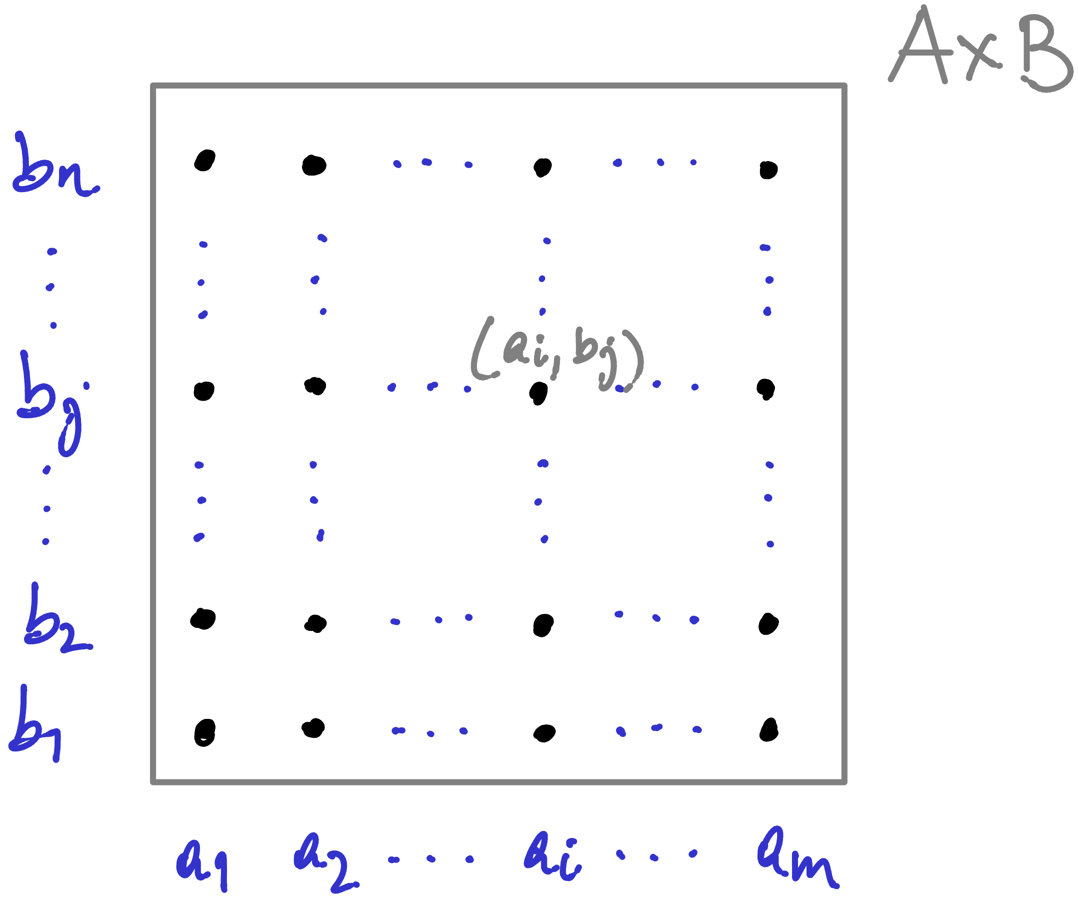
$$\{x \in A \times B \mid \exists a \in A.\, \exists b \in B.\; x = (a,b) \wedge P(a,b)\}.$$

Proposition 89 *For all finite sets* A *and* B,

$$\#(A \times B) = \#A \cdot \#B \quad .$$

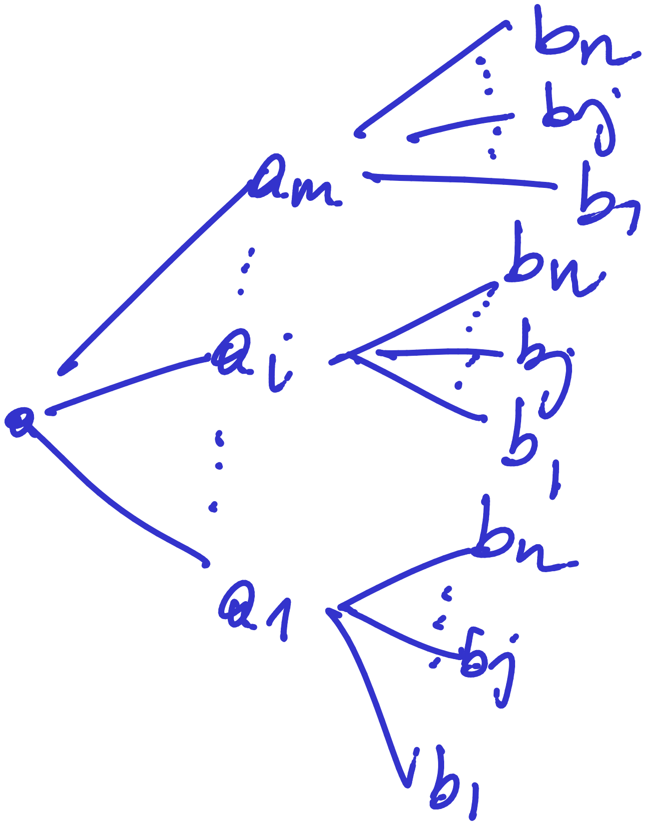
PROOF IDEA: Given $A = \{a_1, \ldots, a_m\}$ $(m \in \mathbb{N})$ and $B = \{b_1, \ldots, b_n\}$ $(n \in \mathbb{N})$ we wish to count

$$
\begin{aligned}
A \times B &= \{ (a, b) \mid a \in A, b \in B \} \\
&= \{ (a_1, b_1), (a_1, b_2), \ldots, (a_1, b_n), \\
&\qquad (a_2, b_1), (a_2, b_2), \ldots, (a_2, b_n), \\
&\qquad \ldots\ , (a_i, b_j), \ldots\ , \\
&\qquad (a_m, b_1), (a_m, b_2), \ldots (a_m, b_n) \}
\end{aligned}
$$



A x B
(ai, bj)
bn
bj
b2
b1
a1 a2 ... ai ... am

So in constructing pairs (a, b) we have $\#A$ choices for a and, for each of these, $\#B$ choices for b.



That is, we have $\#A \cdot \#B$ in total. $\boxtimes$