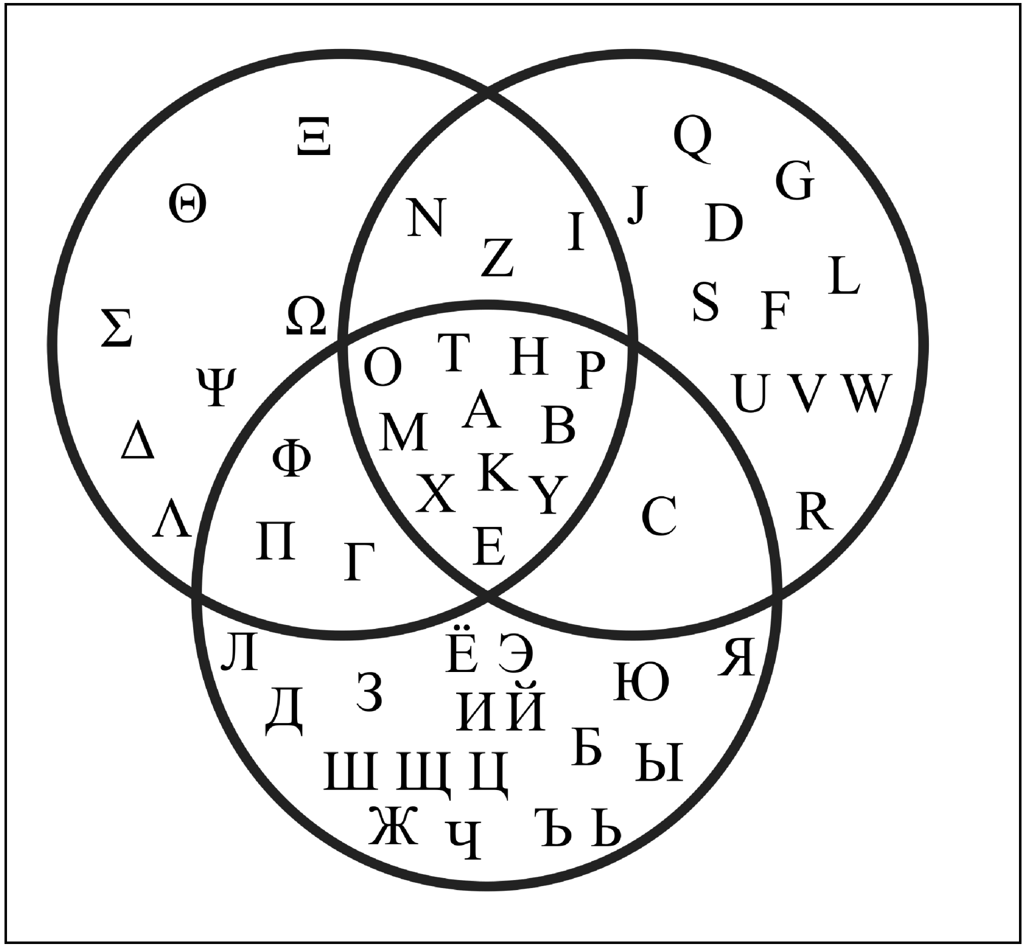
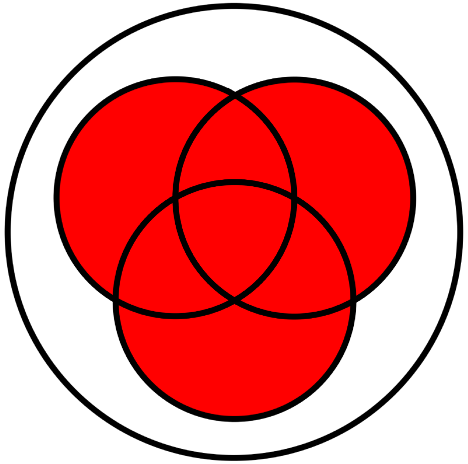
Venn diagrams[a]

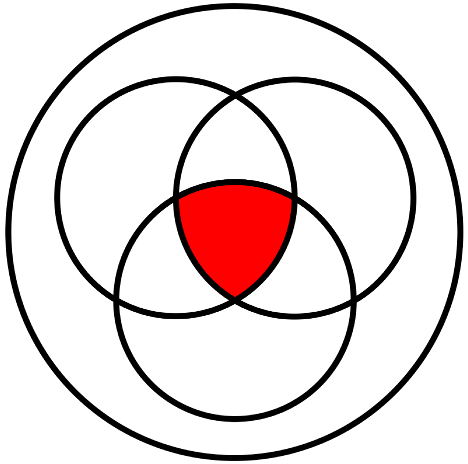


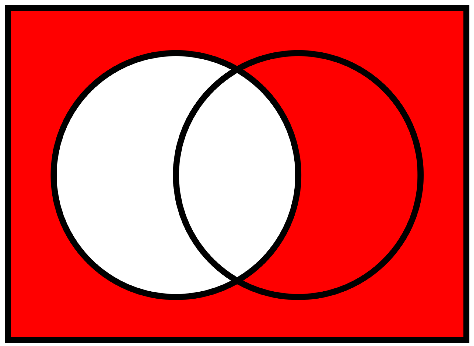
[a]From http://en.wikipedia.org/wiki/Intersection_(set_theory) .

Union



Intersection





Complement

The powerset Boolean algebra

$$(\ \mathcal{P}(U)\ ,\ \ \emptyset\ ,\ \ U\ ,\ \ \cup\ ,\ \ \cap\ ,\ \ (\cdot)^c\ )$$

For all $A, B \in \mathcal{P}(U)$,

$$A \boxed{\cup} B = \{x \in U \mid x \in A \boxed{\vee} x \in B\} \in \mathcal{P}(U)$$

$$A \boxed{\cap} B = \{x \in U \mid x \in A \boxed{\wedge} x \in B\} \in \mathcal{P}(U)$$

$$A^{\boxed{c}} = \{x \in U \mid \boxed{\neg}(x \in A)\} \in \mathcal{P}(U)$$

- The union operation $\cup$ and the intersection operation $\cap$ are associative, commutative, and idempotent.

$$(A \cup B) \cup C = A \cup (B \cup C) \ , \quad A \cup B = B \cup A \ , \quad A \cup A = A$$

$$(A \cap B) \cap C = A \cap (B \cap C) \ , \quad A \cap B = B \cap A \ , \quad A \cap A = A$$

- The *empty set* $\emptyset$ is a neutral element for $\cup$ and the *universal set* $\mathbb{U}$ is a neutral element for $\cap$.

$$\emptyset \cup A \ = \ A \ = \ \mathbb{U} \cap A$$

- The empty set $\emptyset$ is an annihilator for $\cap$ and the universal set $\mathcal{U}$ is an annihilator for $\cup$.

$$\emptyset \cap A = \emptyset$$

$$\mathcal{U} \cup A = \mathcal{U}$$

- With respect to each other, the union operation $\cup$ and the intersection operation $\cap$ are distributive and absorptive.

$$A\cap(B\cup C) = (A\cap B)\cup(A\cap C) \quad , \quad A\cup(B\cap C) = (A\cup B)\cap(A\cup C)$$

$$A \cup (A \cap B) \;=\; A \;=\; A \cap (A \cup B)$$

NB. For all A and B in $\mathcal{P}(\mathcal{U})$,

$$A \cup (A \cap B) = A.$$

PROOF: Let A and B be subsets of $\mathcal{U}$.

RTP: $\forall x \in \mathcal{U}.\ [(x \in A) \vee (x \in A \wedge x \in B)] \Leftrightarrow (x \in A)$.

Let $x \in \mathcal{U}$.

$(\Leftarrow)$ Assume $x \in A$. Then $(x \in A) \vee (x \in A \wedge x \in B)$

$(\Rightarrow)$ Assume $(x \in A) \vee (x \in A \wedge x \in B)$.

CASE $(x \in A)$: Then $(x \in A)$.

CASE $(x \in A \wedge x \in B)$: Then $(x \in A)$. $\boxtimes$

- The complement operation $(\cdot)^c$ satisfies complementation laws.

$$A \cup A^c = \mathcal{U} \ , \quad A \cap A^c = \emptyset$$

NB: For $A \in \mathcal{P}(\mathcal{U})$, $A \cap A^c = \emptyset$

PROOF: Let A be a subset of $\mathcal{U}$.

<u>RTP</u>: $\forall x \in \mathcal{U}. \left[x \in A \wedge \neg(x \in A) \right] \iff$ <u>false</u>

Let $x \in \mathcal{U}$.

($\Longrightarrow$) Assume $x \in A$ and $(x \in A) \Rightarrow$ <u>false</u>.
Then, by MP, we have <u>false</u> as required.

($\Longleftarrow$) Vacuously. $\boxtimes$

Proposition 85 *Let* $\mathcal{U}$ *be a set and let* $A, B \in \mathcal{P}(\mathcal{U})$.

1. $\forall X \in \mathcal{P}(\mathcal{U}).\ A \cup B \subseteq X \iff (A \subseteq X \wedge B \subseteq X)$.

2. $\forall X \in \mathcal{P}(\mathcal{U}).\ X \subseteq A \cap B \iff (X \subseteq A \wedge X \subseteq B)$.

PROOF:

(1) Let $\mathcal{U}$ be a set and let $A \subseteq \mathcal{U}$, $B \subseteq \mathcal{U}$.

Consider $X \subseteq \mathcal{U}$.

($\Rightarrow$) Assume

$$\forall z \in \mathcal{U}.\ (z \in A \vee z \in B) \Rightarrow z \in X \qquad (*)$$

RTP (i) $\forall x \in \mathcal{U}.\ x \in A \Rightarrow x \in X$

and (ii) $\forall y \in \mathcal{U}.\ y \in B \Rightarrow y \in X$

RTP (i) $\forall x \in U.\ x \in A \Rightarrow x \in X$.

Let $x \in U$ be arbitrary

Assume: $x \in A$ $(**)$

RTP: $x \in X$

By $(*)$, by instantiation we have

$$x \in A \vee x \in B \Rightarrow x \in X \qquad (1)$$

By $(**)$, we have

$$x \in A \vee x \in B \qquad (2)$$

From (1) and (2), $x \in X$ follows.

<u>RTP</u> (ii) $\forall y \in U.\ y \in B \Rightarrow y \in X$

Analogous and left as an exercise.

($\Leftarrow$) <u>Assume</u>

① $\forall x \in U.\ x \in A \Rightarrow x \in X$

and

② $\forall y \in U\ y \in B \Rightarrow y \in X$.

<u>RTP</u>: $\forall z \in U.\ (z \in A \vee z \in B) \Rightarrow z \in X$.

Let $z \in U$ be arbitrary.

<u>Assume</u>

(*) $(z \in A) \vee (z \in B)$

<u>RTP</u> $z \in X$

Instantiating ①, we have

$$z \in A \Rightarrow z \in X \qquad ②$$

and, instantiating ②, we have

$$z \in B \Rightarrow z \in X \qquad ③$$

Using assumption (*), we need show

$z \in X$ under two cases:

<u>CASE</u> if $z \in A$ Then by ②, we are done.

<u>CASE</u> if $z \in B$ Then by ③, we are done.

$\boxtimes$

Corollary 86 *Let* U *be a set and let* $A, B, C \in \mathcal{P}(U)$.

1. $C = A \cup B$

iff

$$[A \subseteq C \wedge B \subseteq C]$$

$$\wedge$$

$$[\forall X \in \mathcal{P}(U).\ (A \subseteq X \ \wedge\ B \subseteq X) \implies C \subseteq X]$$

2. $C = A \cap B$

iff

$$[C \subseteq A \wedge C \subseteq B]$$

$$\wedge$$

$$[\forall X \in \mathcal{P}(U).\ (X \subseteq A \ \wedge\ X \subseteq B) \implies X \subseteq C]$$

Sets and logic

$\mathcal{P}(\mathcal{U})$	$\{\,\mathbf{false}\,,\ \mathbf{true}\,\}$
$\emptyset$	$\mathbf{false}$
$\mathcal{U}$	$\mathbf{true}$
$\cup$	$\vee$
$\cap$	$\wedge$
$(\cdot)^{\mathrm{c}}$	$\neg(\cdot)$