N.B. for all U,

$$\emptyset \in \mathcal{P}(U), \quad U \in \mathcal{P}(U).$$

Powerset axiom

For any set, there is a set consisting of all its subsets.

$$\mathcal{P}(U) \stackrel{\text{def}}{=} \{ X \mid X \subseteq U \}$$

$$\forall X.\ X \in \mathcal{P}(U) \iff X \subseteq U \quad .$$

Example:

$$\mathcal{P}(\{x,y,z\})$$ subsets of cardinality

$$= \{ \emptyset, \qquad 0$$

$$\{x\}, \{y\}, \{z\}, \qquad 1$$

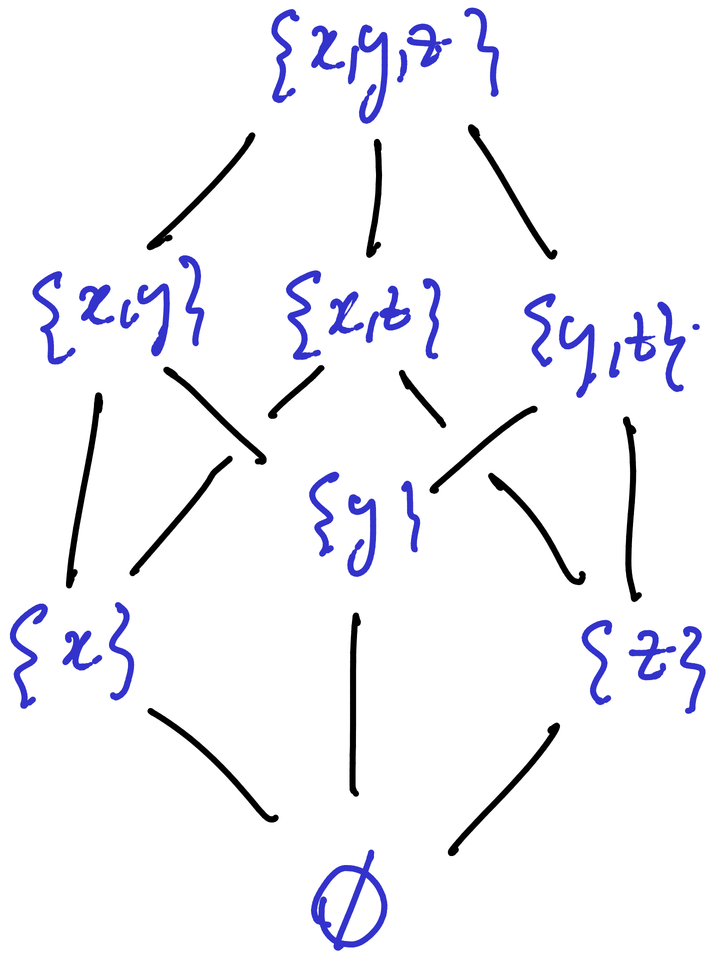
$$\{x,y\}, \{x,z\}, \{y,z\}, \qquad 2$$

$$\{x,y,z\} \} \qquad 3$$

$$\# \mathcal{P}(\{x,y,z\}) = 8$$

Hasse diagrams

$$\mathcal{P}(\{x,y,z\}) =$$



Proposition 84 *For all finite sets* $\mathcal{U}$,

$$\#\,\mathcal{P}(\mathcal{U}) = 2^{\#\mathcal{U}} \quad .$$

PROOF IDEA:

(1) For a set $\mathcal{U}$ and a natural number k,
define $\mathcal{P}^{(k)}(\mathcal{U}) \stackrel{\text{def}}{=} \{ X \mid X \subseteq \mathcal{U} \wedge \#X = k \}$

For instance, $\mathcal{P}^{(0)}(\mathcal{U}) = \{ \emptyset \}$.

Then $\#\,\mathcal{P}^{(k)}(\mathcal{U}) = \binom{\#\mathcal{U}}{k}$

and

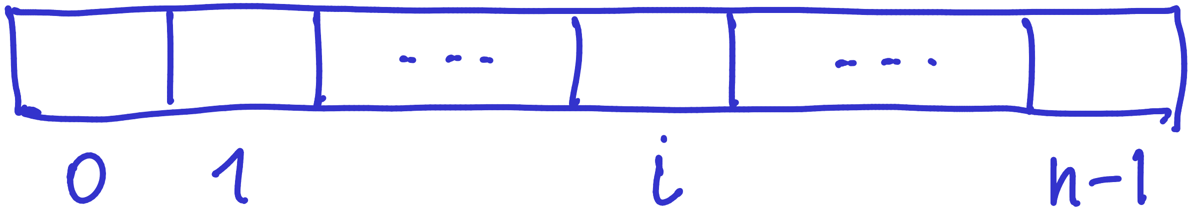
$$\#\,\mathcal{P}(\mathcal{U}) = \sum_{k=0}^{\#\mathcal{U}} \#\,\mathcal{P}^{(k)}(\mathcal{U}) = \sum_{k=0}^{\#\mathcal{U}} \binom{\#\mathcal{U}}{k} = 2^{\#\mathcal{U}} \quad \boxtimes$$

(2) • For $n \in \mathbb{N}$, $[n] \stackrel{\text{def}}{=} \{0, 1, \ldots, n-1\}$

• $\# \mathcal{P}([n])$?

• $S \in \mathcal{P}([n])$ is determined by whether or not $i \in S$ for $i = 0, \ldots, n-1$

• Consider boolean (or bit) valued arrays

$s =$ 

as encoding subsets S of $[n]$, with $s(i) = \text{true}$ iff $i \in S$.

For instance,

- the empty subset is encoded by

false	false	...	false	...	false
0	1		i		$n-1$

- the singleton subset $\{i\}$ is encoded by

false	false	...	true	...	false
0	1		i		$n-1$

- the subset of odd numbers is encoded by

false	...	false	true	false	...
0			$i=2k+1$		$n-1$

- We need count the number of boolean (or bit) valued arrays of size n;
- equivalently the sequences (or strings) of bits of length n.
- Each of these corresponds to the binary representation of a natural number below 2^n.
- Hence we have a total of 2^n.

$\boxtimes$

NB: The powerset construction can be iterated.

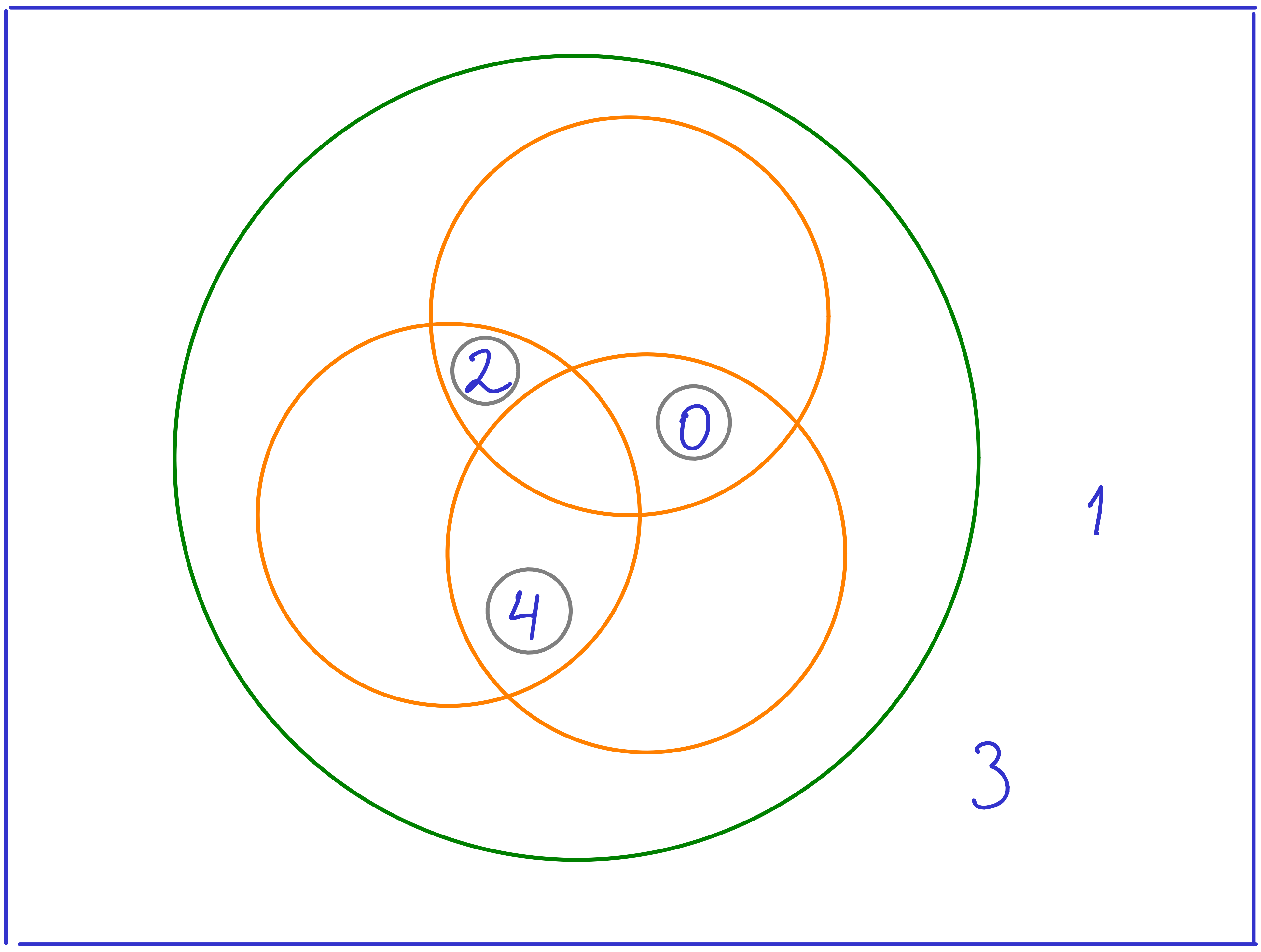
In particular,

$$\mathcal{F} \in \mathcal{P}(\mathcal{P}(\mathcal{U})) \iff \mathcal{F} \subseteq \mathcal{P}(\mathcal{U})$$

That is, $\mathcal{F}$ is a set of subsets of $\mathcal{U}$, sometimes referred to as a family.

Example: The family $\mathcal{E} \subseteq \mathcal{P}([5])$ consisting of the non-empty subsets of $[5] \stackrel{\text{def}}{=} \{0,1,2,3,4\}$ all whose elements are even is

$$\mathcal{E} = \Big\{ \{0\}, \{2\}, \{4\}, \{0,2\}, \{0,4\}, \{2,4\}, \{0,2,4\} \Big\}$$


2
0
4
1
3

Exercise: Explicitly describe the family

$$\mathcal{S} = \left\{ S \subseteq [5] \;\middle|\; \begin{array}{l} \text{the sum of the elements} \\ \text{of } S \text{ is } 6 \end{array} \right\}$$

and depict its Hasse and Venn diagrams.