Powerset axiom

N.B. for all \( U \),

\[ \emptyset \in \mathcal{P}(U), \ U \in \mathcal{P}(U). \]

For any set, there is a set consisting of all its subsets.

\[ \mathcal{P}(U) \overset{\text{def}}{=} \{ x \mid x \subseteq U \} \]

\[ \forall X. \ X \in \mathcal{P}(U) \iff X \subseteq U. \]
Example:

$$\mathcal{P}(\{x, y, z\}) = \{\emptyset, \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}, \{x, y, z\}\}$$

subsets of cardinality

$$\#\mathcal{P}(\{x, y, z\}) = 8$$
\( \mathcal{P}(\{x, y, z\}) = \)
Proposition 84  For all finite sets $U$, 

$$\# \mathcal{P}(U) = 2^{\# U}.$$ 

Proof Idea:

1. For a set $U$ and a natural number $k$, define $\mathcal{P}^{(k)}(U) = \{ X \mid X \subseteq U \wedge \# X = k \}$.

For instance, $\mathcal{P}^{(0)}(U) = \{ \emptyset \}$.

Then $\# \mathcal{P}^{(k)}(U) = \binom{\# U}{k}$ and

$$\# \mathcal{P}(U) = \sum_{k=0}^{\# U} \# \mathcal{P}^{(k)}(U) = \sum_{k=0}^{\# U} \binom{\# U}{k} = 2^{\# U}.$$

$\Box$
(2) \textbullet \text{ For } n \in \mathbb{N}, \quad [n] = \{0, 1, \ldots, n-1\}\\
\textbullet \quad \#P([n]) \text{?}\\
\textbullet \quad S \in P([n]) \text{ is determined by whether or not } i \in S \text{ for } i = 0, \ldots, n-1\\
\textbullet \quad \text{Consider boolean (or bit) valued arrays } S = \begin{array}{cccc}
\hline
& & & \\
0 & 1 & i & n-1 \\
\hline
\end{array}\\
as \text{ encoding subsets } S \text{ of } [n], \text{ with } S(i) = \text{true if and only if } i \in S.
For instance,

- the empty subset is encoded by

\[
\begin{array}{cccc}
\text{false} & \text{false} & \ldots & \text{false} \\
0 & 1 & i & n-1
\end{array}
\]

- the singleton subset \( \{i, i\} \) is encoded by

\[
\begin{array}{cccc}
\text{false} & \text{false} & \ldots & \text{true} & \ldots & \text{false} \\
0 & 1 & i & n-1
\end{array}
\]

- the subset of odd numbers is encoded by

\[
\begin{array}{cccc}
\text{false} & \ldots & \text{false} & \text{true} & \text{false} & \ldots \\
0 & i=2k+1 & n-1
\end{array}
\]
• We need count the number of boolean (or bit) valued arrays of size \( n \);
• equivalently the sequences (or strings) of bits of length \( n \).
• Each of these corresponds to the binary representation of a natural number below \( 2^n \).
• Hence we have a total of \( 2^n \).
NB: The powerset construction can be iterated. In particular,

\[F \in \mathcal{P}(\mathcal{P}(U)) \iff F \subseteq \mathcal{P}(U)\]

That is, \(F\) is a set of subsets of \(U\), sometimes referred to as a family.
Example: The family $\mathcal{E} \subseteq P([5])$ consisting of the non-empty subsets of $[5] = \{0, 1, 2, 3, 4\}$ all whose elements are even is

$$\mathcal{E} = \left\{ \emptyset, \{2\}, \{4\}, \{0, 2\}, \{0, 4\}, \{2, 4\}, \{0, 2, 4\} \right\}$$
Exercise: Explicitly describe the family

\[ S = \{ S \subseteq \{5\} \mid \text{the sum of the elements of } S \text{ is } 6 \} \]

and depict its Hasse and Venn diagrams.